TRINITY UNIVERSITY

How to solve an Equation

A Topological Approach

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A Basic Math Problem

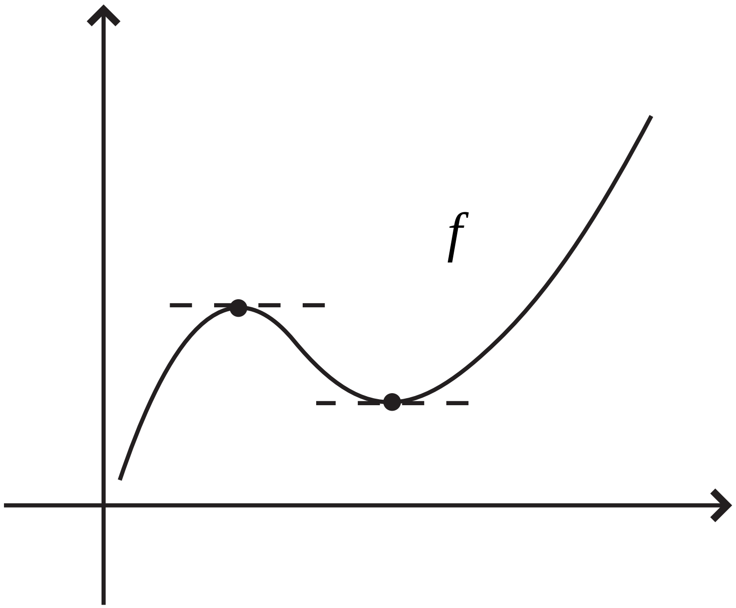
Solve

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ f(x) &= 0 \end{aligned}$$

Questions:

- Are there solutions? (Is f **surjective** (or **onto**)?)
- Is the solution unique? (Is f **injective**)?
- Can we decide if a map f is invertible (**bijective**)?

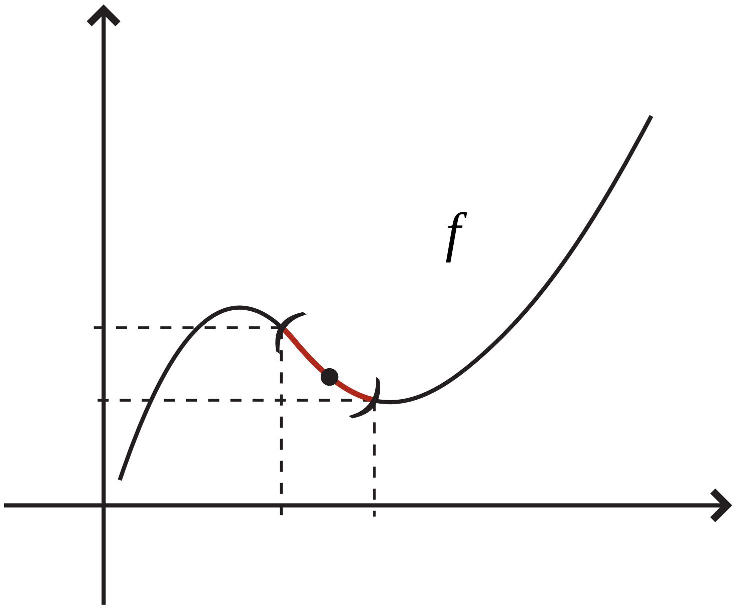
Basic Principles



- f is not invertible (not 1-1).
- There are points where $f'(x) = 0$.

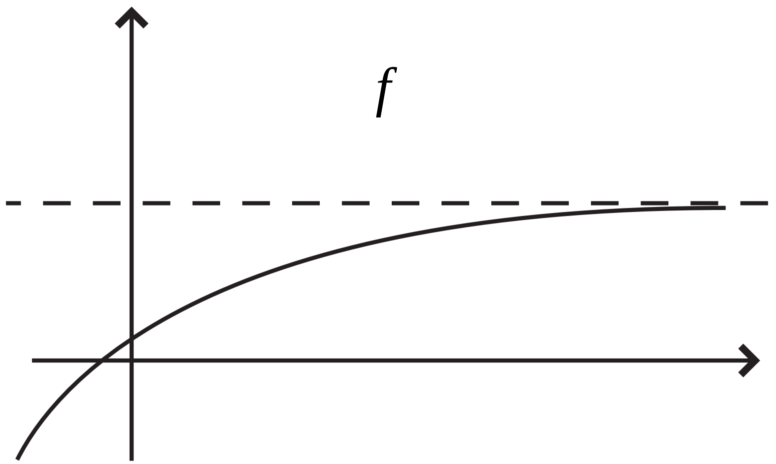
First Condition: The map f must not have any Critical Points.

Local vs. Global



- Local: Consider a solution *near* a point.
- Global: Consider a solution in the *entire* domain.
  - ▶ Is Local invertibility sufficient for Global invertibility?

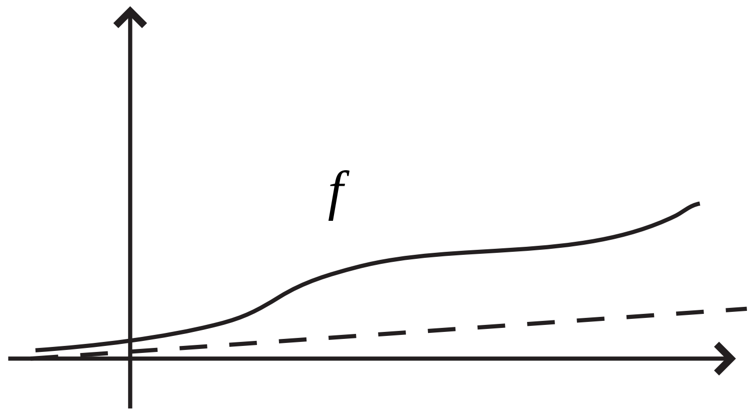
Basic Principles



- $f'(x) \neq 0$, however f is not invertible (not onto).
- $f'(x) \to 0$ as $x \to \infty$.

Basic Principles

Idea: What if $|f'(x)| \geq \delta > 0$?



Theorem: If $|f'(x)| \geq \delta > 0$, then $f(x) = a$ has *exactly* one solution for every $a \in \mathbb{R}$, that is, f is bijective.

Basic Principles

In general, we would like to solve a system of equations:

$$\begin{cases} 3x + 2y = 5 \\ 2x + y = 1 \end{cases} \leadsto \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$A\overrightarrow{v} = \overrightarrow{b}$$

$$\overrightarrow{v} = A^{-1}\overrightarrow{b} \leadsto \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

Basic Principles

Recall the **Jacobian Matrix** of a map

$$
\begin{aligned}
F: \quad & \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\
& (x,y) \mapsto \big(f(x,y), g(x,y)\big)
\end{aligned}
$$

$$
J_F = DF(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix}
$$

▶ Jacobian is the generalization of derivative.

Basic Result

Theorem (Gale-Nikaido, 72). Let $F:\mathbb{R}^2\to\mathbb{R}^2$ given by $F(x,y)=\big(f(x,y),g(x,y)\big)$. If there are positive numbers m,n and M,N such that

$$m\le|f_x|\le M$$
$$n\le|f_xg_y-f_yg_x|\le N,$$

then the system of equations

$$f(x,y)=a$$
$$g(x,y)=b,$$

has exactly one solution for any a,b. In other words, F is bijective.

Basic Result

Proof: Fix a and y. Then $f(x,y) = a$ as a function of x has a solution $x = \varphi(y)$, that is, $f(\varphi(y), y) = a$.

- Ex: $f(x,y) = x + y$, then $x = \varphi(y) = a - y$

Let $G(y) = g(\varphi(y), y)$, we must solve $G(y) = b$; hence estimate $G'(y)$

(Chain rule) $G'(y) = g_x(\varphi(y), y)\varphi'(y) + g_y(\varphi(y), y) = g_x\varphi' + g_y$

To find φ': $f(\varphi(y), y) = a \Longrightarrow f_x\varphi' + f_y = 0 \Longrightarrow \varphi' = -\dfrac{f_y}{f_x}$

$$G'(y) = g_x\left(\frac{-f_y}{f_x}\right) + g_y$$

$$|G'(y)| = \frac{|f_x g_y - f_x g_y|}{|f_x|} \Longrightarrow \frac{n}{M} \le |G'(y)| \le \frac{N}{m}$$

Thus G is invertible.

Higher Dimensional Results

A smooth map $(x_1, \ldots, x_n) \overset{f}{\longmapsto} (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ is a **local diffeomorphism** if the **Jacobian Matrix**

$$Df(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_1}(x) & & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix} \text{ is invertible.}$$

Goal:

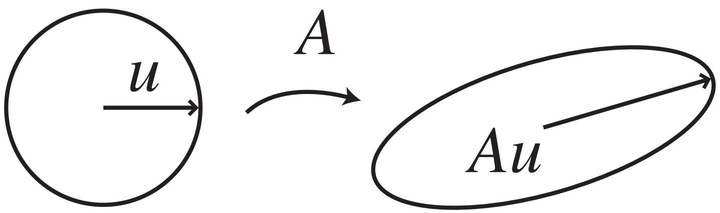
A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if $\underset{\text{(condition)}}{\underline{\hspace{3cm}}}$

- Motivated by elementary discussion, look at "**size**" of $Df(x)$.

Norm of Matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{pmatrix}$$, can we measure the **size** (norm) of A?

- $\|A\| = \sqrt{a_{11}^2 + \cdots + a_{1n}^2 + \cdots + a_{n1}^2 + \cdots + a_{nn}^2}$.
- $\|A\| = \max |a_{ij}|$.
- $\|A\| = \max \|Au\|$, u is a n-vector of norm 1.



Fact: All the norms above are **equivalent**.

Classical Result

Theorem (Hadamard-Plastock, 74). A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if

$$\|Df(x)^{-1}\|^{-1} \geq \delta > 0.$$

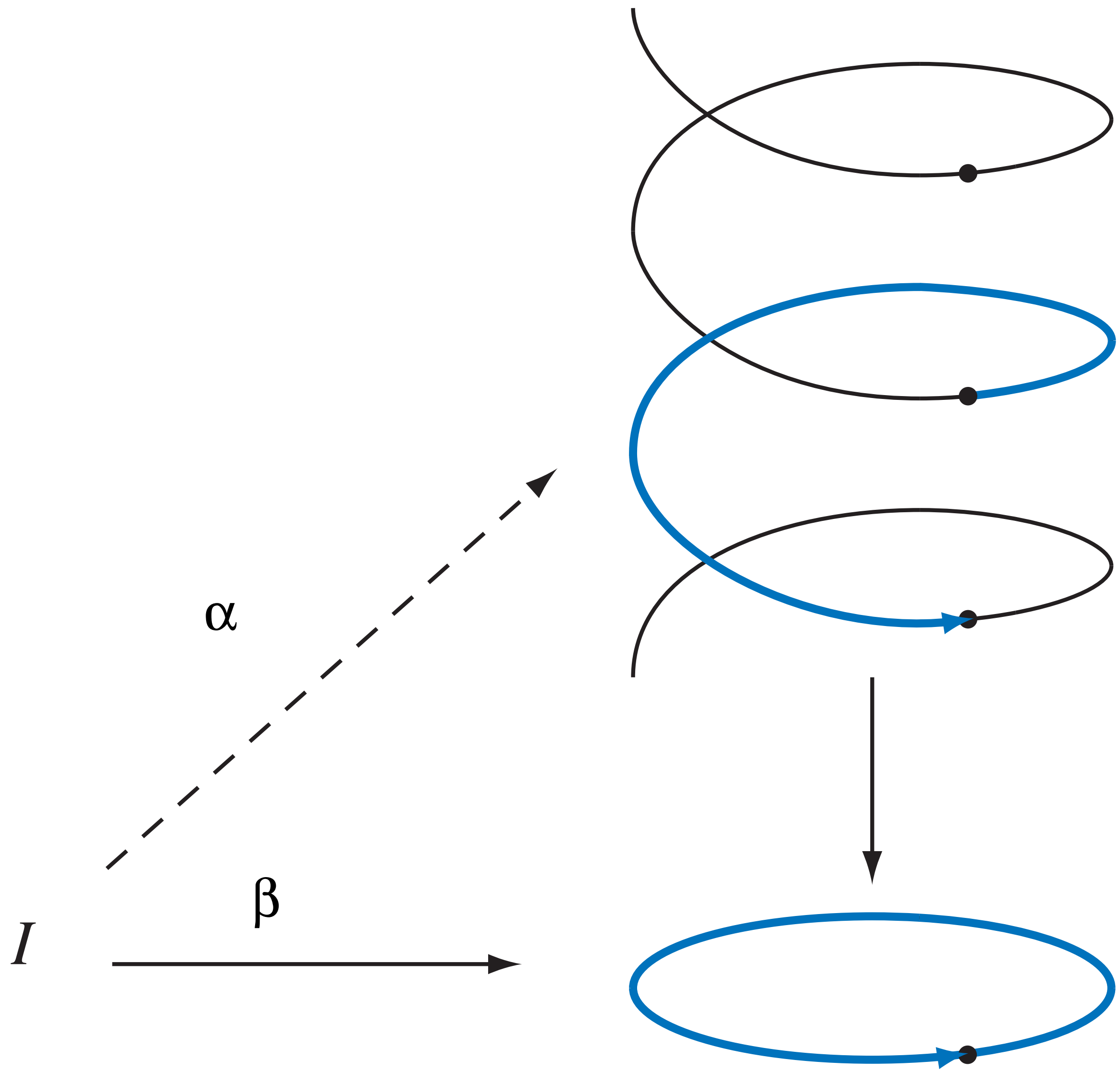
In one dimension, $f : \mathbb{R}^1 \to \mathbb{R}^1$, $Df(x) = f'(x)$.

$$\|Df(x)^{-1}\|^{-1} = \left|\frac{1}{f'(x)}\right|^{-1} = |f'(x)|$$

Idea: Show that f is a *covering map*.

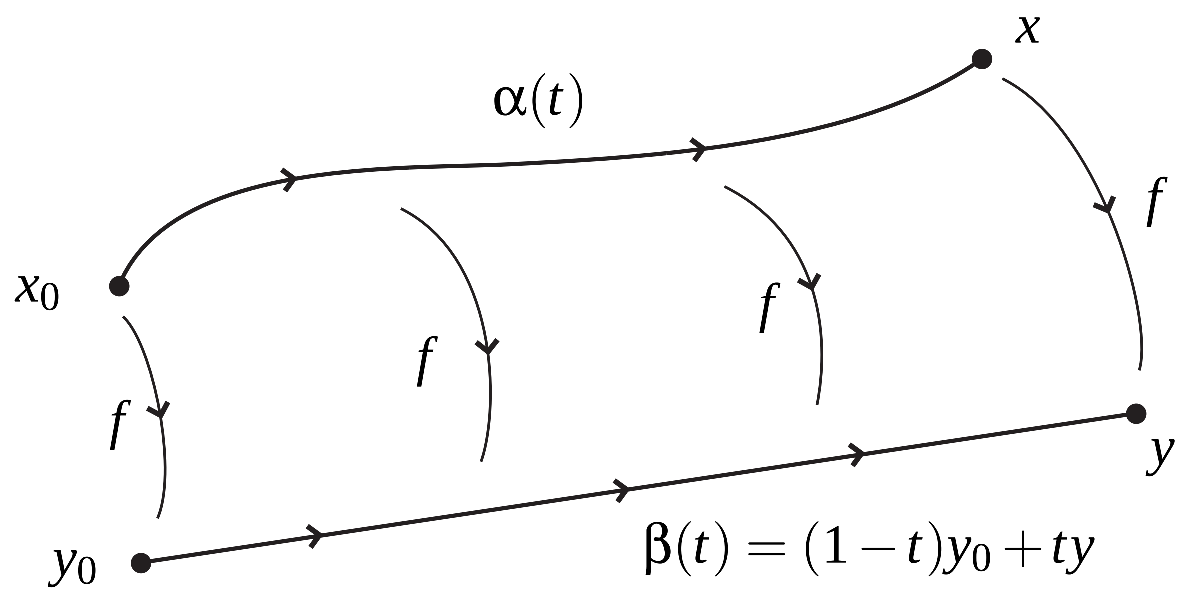
Proposition. A covering map $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective.

Covering Maps



Property: f is a Covering Map $\iff$ f lifts paths.

Lift of Line Segments



Solve for α: $\quad f(\alpha(t)) = \beta(t) = (1-t)y_0 + ty$, $t \in [0,1]$.

$$Df(\alpha(t)) \cdot \alpha'(t) = y - y_0$$

$$(*) \begin{cases} \alpha'(t) &= \left[Df\left(\alpha(t)\right)\right]^{-1}(y-y_0) \\ \alpha(0) &= x_0 \end{cases}$$, for all $t \in [0,1]$.

- Condition on $\left\| \left[Df(x)\right]^{-1} \right\|^{-1}$ implies **global** solution of $(*)$.

Topological Result

Hadamard Theorem: Analysis **detects** Topology.

Topology: Study of shapes

- properties preserved under continuous deformations

▶ Basic objects:

- Open sets: open intervals (e.g. $(0,1)$), open balls
  $x \in A$, $\exists\, \varepsilon > 0$, $B(x,\varepsilon) \subset A$.

- Closed sets: closed intervals (e.g $[0,1]$), closed balls
  $x_i \in F,\, x_i \longrightarrow x \Longrightarrow x \in F$

- Compact sets: closed and bounded.

Topological Result

- Def: A map $f : X \longrightarrow X$ is **proper** if the pre-image of compact sets is compact.

Analytically; $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is proper if and only if

$$\lim_{|x| \to \infty} \|f(x)\| = \infty$$

Hadarmard's Theorem. *A local homeomorphism is* ***bijective*** *if and only if it is* ***proper****.*

Top. Result - Hadamard

▶ Idea of proof:

Take $f(x_0) = y_0$ and choose $y \in \mathbb{R}^n$

Local homeomorphism $\Rightarrow$ Exist $x(t)$ for $\varepsilon > 0$ s.t.

$$f(x(t)) = ty + (1-t)y_0, \text{ for } t \in [0, \varepsilon)$$

Extend $x(t)$ as much as possible, say to $0 < \beta < \infty$.

Take $t_i \longrightarrow \beta$ and $f^{-1}(\{ty + (1-t)y_0, t \in [0,1]\})$ is compact.

- (Bolzano-Weierstrass) $\Rightarrow x(t_i) \longrightarrow \overline{x}$ as $t_i \longrightarrow \beta$
- (Continuity) $\Rightarrow f(\overline{x}) = \beta y + (1-\beta)y_0$, can be extended past β.

Variational Calculus

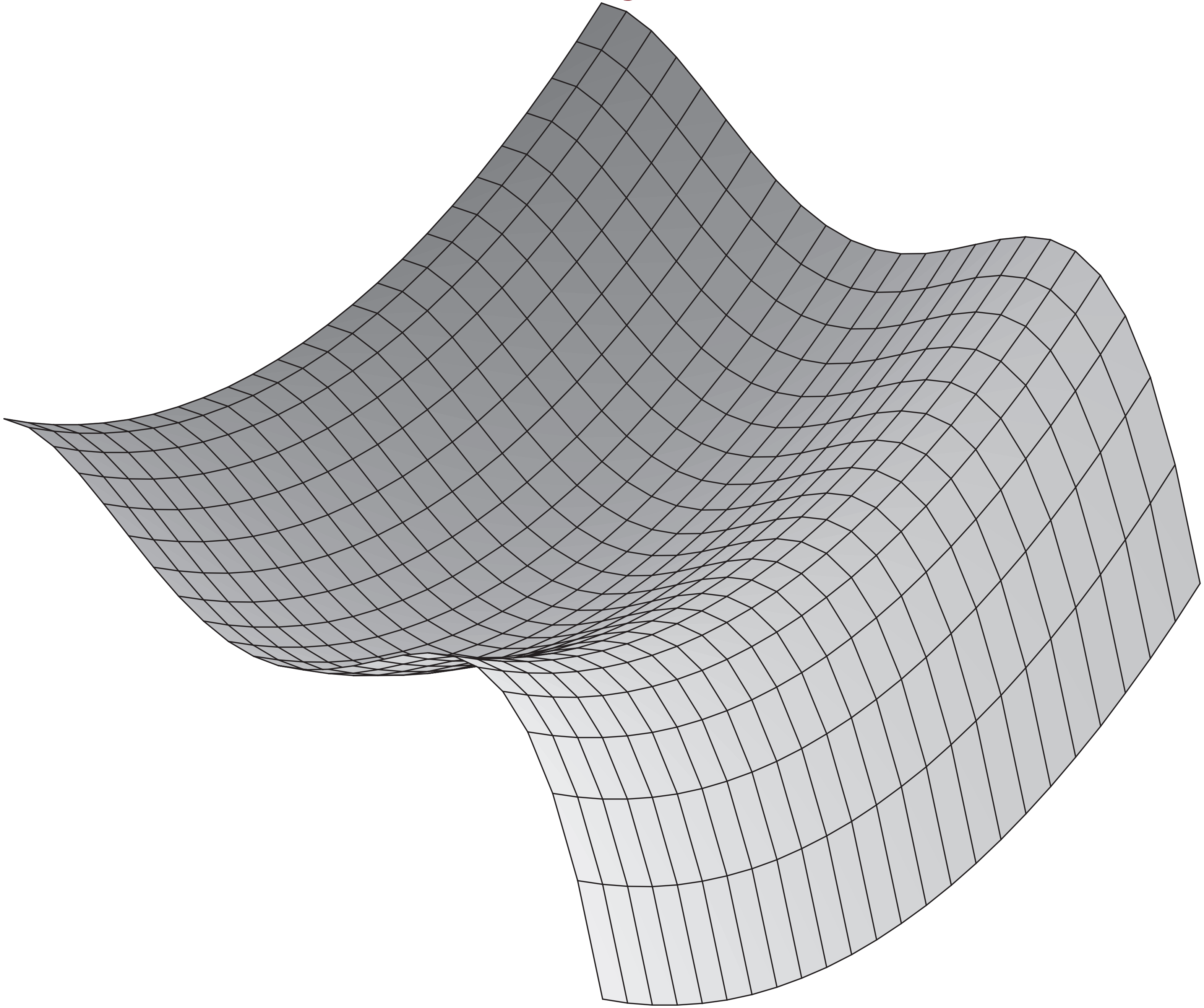
▶ Method in Analysis (PDE) to find **Critical Points** of Functionals
($F : X \to \mathbb{R}$)

- Idea to solve $f(x) = 0$. If we knew that $f = F'$, then the critical points of F would be a solution to $f(x) = 0$.

- Theorem of Mountain Pass Type

Rolle's Theorem. *Let $f \in C^1([x_1, x_2], \mathbb{R})$. If $f(x_1) = f(x_2)$, then there exists $x_3 \in (x_1, x_2)$ such that $f'(x_3) = 0$.*

Mountain Pass Geometry



Mountain Pass Theorem

Theorem (Finite Dimensional MPT, Courant). *Suppose that a continuous function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is proper and possesses two distinct strict relative minima x_1 and x_2. Then F possesses a third critical point x_3 distinct from x_1 and x_2, characterized by*

$$F(x_3) = \inf_{\Sigma \in \Gamma} \max_{x \in \Sigma} F(x)$$

where $\Gamma = \{\Sigma \subset \mathbb{R}^n; \Sigma$ is compact and connected and $x_1, x_2 \in \Sigma\}$. Moreover, x_3 is not a relative minimizer, that it, in every neighborhood of x_3 there exists a point x such that $F(x) < F(x_3)$.

Application of MPT

Hadarmard's Theorem. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a local diffeomorphism so that* $\|f(x)\| \to \infty$ *as* $\|x\| \to \infty$, *then* f *is a diffeomorphism.*

Sketch of Proof: (show that f is onto and injective)

- Check that f is onto.

- Injectivity by contradiction.

Suppose $f(x_1) = f(x_2) = y$, then define $F(x) = \frac{1}{2}\|f(x) - y\|^2$

Check the MPT geometry $\Rightarrow \exists x_3$, $F(x_3) > 0$ (i.e., $\|f(x_3) - y\| > 0$.), but

$$F'(x_3) = \nabla^T f(x_3) \cdot (f(x_3) - y) = 0$$