How to solve an Equation

A Topological Approach

E. Cabral Balreira
Mathematics Department
Trinity University
San Antonio, TX 78212
ebalreira@nd.edu
http://www.trinity.edu/ebalreir
A Basic Math Problem

Solve

\[ x^2 - 5x + 6 = 0 \]
\[ f(x) = 0 \]

Questions:

- Are there solutions? (Is \( f \) surjective (or onto)?)
- Is the solution unique? (Is \( f \) injective)?
- Can we decide if a map \( f \) is invertible (bijective)?
Basic Principles

- $f$ is not invertible (not 1-1).
- There are points where $f'(x) = 0$.

First Condition: The map $f$ must not have any Critical Points.
Local vs. Global

- Local: Consider a solution near a point.

- Global: Consider a solution in the entire domain.

▶ Is Local invertibility sufficient for Global invertibility?
Basic Principles

- $f'(x) \neq 0$, however $f$ is not invertible (not onto).

- $f'(x) \to 0$ as $x \to \infty$. 
Basic Principles

Idea: What if $|f'(x)| \geq \delta > 0$?

**Theorem:** If $|f'(x)| \geq \delta > 0$, then $f(x) = a$ has exactly one solution for every $a \in \mathbb{R}$, that is, $f$ is bijective.
Basic Principles

In general, we would like to solve a system of equations:

\[
\begin{cases}
3x + 2y = 5 \\
2x + y = 1
\end{cases}
\Rightarrow
\begin{bmatrix}
3 & 2 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
5 \\
1
\end{bmatrix}
\]

\[\vec{v} = A^{-1} \vec{b}\]

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
-1 & 2 \\
2 & 3
\end{bmatrix}
\begin{bmatrix}
5 \\
1
\end{bmatrix} =
\begin{bmatrix}
-3 \\
7
\end{bmatrix}
\]
Recall the **Jacobian Matrix** of a map

\[ F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \]

\[ (x, y) \mapsto (f(x, y), g(x, y)) \]

\[ J_F = DF(x, y) = \begin{pmatrix}
\frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\
\frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y)
\end{pmatrix} \]

► Jacobian is the generalization of derivative.
Theorem (Gale-Nikaido, 72). Let \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( F(x, y) = (f(x, y), g(x, y)) \). If there are positive numbers \( m, n \) and \( M, N \) such that

\[
\begin{align*}
    m &\leq |f_x| \leq M \\
n &\leq |f_x g_y - f_y g_x| \leq N,
\end{align*}
\]

then the system of equations

\[
\begin{align*}
f(x, y) &= a \\
g(x, y) &= b,
\end{align*}
\]

has exactly one solution for any \( a, b \). In other words, \( F \) is bijective.
**Basic Result**

Proof: Fix $a$ and $y$. Then $f(x,y) = a$ as a function of $x$ has a solution $x = \varphi(y)$, that is, $f(\varphi(y), y) = a$.

- Ex: $f(x,y) = x + y$, then $x = \varphi(y) = a - y$

Let $G(y) = g(\varphi(y), y)$, we must solve $G(y) = b$; hence estimate $G'(y)$

(Chain rule) $G'(y) = g_x(\varphi(y), y)\varphi'(y) + g_y(\varphi(y), y) = g_x\varphi' + g_y$

To find $\varphi'$: $f(\varphi(y), y) = a \implies f_x\varphi' + f_y = 0 \implies \varphi' = -\frac{f_y}{f_x}$

$$G'(y) = g_x \left( -\frac{f_y}{f_x} \right) + g_y$$

$$|G'(y)| = \frac{|f_xg_y - f_xg_y|}{|f_x|} \implies \frac{n}{M} \leq |G'(y)| \leq \frac{N}{m}$$

Thus $G$ is invertible.
Higher Dimensional Results

A smooth map \((x_1, \ldots, x_n) \overset{f}{\rightarrow} (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))\) is a **local diffeomorphism** if the **Jacobian Matrix**

\[
Df(x) := \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x)
\end{pmatrix}
\]

is invertible.

Goal:

A local diffeomorphism \(f : \mathbb{R}^n \to \mathbb{R}^n\) is bijective if \(\text{condition}\)

\[\blacktriangleright\] Motivated by elementary discussion, look at “**size**” of \(Df(x)\).
Norm of Matrices

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\]

, can we measure the size (norm) of \( A \)?

- \[ \|A\| = \sqrt{a_{11}^2 + \cdots + a_{1n}^2 + \cdots + a_{n1}^2 + \cdots + a_{nn}^2}. \]
- \[ \|A\| = \max |a_{ij}|. \]
- \[ \|A\| = \max \|Au\|, \text{ } u \text{ is a } n \text{-vector of norm 1.} \]

Fact: All the norms above are equivalent.
Classical Result

Theorem (Hadamard-Plastock, 74). A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if

$$\|Df(x)^{-1}\|^{-1} \geq \delta > 0.$$  

In one dimension, $f : \mathbb{R}^1 \to \mathbb{R}^1, Df(x) = f'(x)$.

$$\|Df(x)^{-1}\|^{-1} = \left| \frac{1}{f'(x)} \right|^{-1} = |f'(x)|$$

Idea: Show that $f$ is a covering map.

Proposition. A covering map $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective.
Property: $f$ is a Covering Map $\iff f$ lifts paths.
Lift of Line Segments

Solve for $\alpha$:

$$f(\alpha(t)) = \beta(t) = (1 - t)y_0 + ty, \quad t \in [0, 1].$$

$$Df(\alpha(t)) \cdot \alpha'(t) = y - y_0$$

\[
\begin{align*}
\alpha'(t) &= \left[Df(\alpha(t))\right]^{-1}(y - y_0), \\
\alpha(0) &= x_0
\end{align*}
\]

- Condition on $\left\|\left[Df(x)\right]^{-1}\right\|^{-1}$ implies **global** solution of $(\ast)$. 
Topological Result

Hadamard Theorem: Analysis detects Topology.

Topology: Study of shapes
- properties preserved under continuous deformations

► Basic objects:

- Open sets: open intervals (e.g. \((0, 1)\)), open balls
  \[ x \in A, \exists \varepsilon > 0, B(x, \varepsilon) \subset A. \]

- Closed sets: closed intervals (e.g. \([0, 1]\)), closed balls
  \[ x_i \in F, x_i \longrightarrow x \longrightarrow x \in F \]

- Compact sets: closed and bounded.
**Topological Result**

- Def: A map $f : X \to X$ is **proper** if the pre-image of compact sets is compact.

Analytically; $f : \mathbb{R}^n \to \mathbb{R}^n$ is proper if and only if

$$\lim_{|x| \to \infty} \|f(x)\| = \infty$$

**Hadamard’s Theorem.** A local homeomorphism is **bijective** if and only if it is **proper**.
Idea of proof:

Take \( f(x_0) = y_0 \) and choose \( y \in \mathbb{R}^n \)

Local homeomorphism \( \Rightarrow \) Exist \( x(t) \) for \( \varepsilon > 0 \) s.t.

\[
f(x(t)) = ty + (1 - t)y_0, \text{ for } t \in [0, \varepsilon)
\]

Extend \( x(t) \) as much as possible, say to \( 0 < \beta < \infty \).

Take \( t_i \to \beta \) and \( f^{-1}(\{ty + (1 - t)y_0, t \in [0, 1]\}) \) is compact.

- (Bolzano-Weierstrass) \( \Rightarrow \) \( x(t_i) \to \bar{x} \) as \( t_i \to \beta \)

- (Continuity) \( \Rightarrow \) \( f(\bar{x}) = \beta y + (1 - \beta)y_0 \), can be extended past \( \beta \).
Variational Calculus

- Method in Analysis (PDE) to find **Critical Points** of Functionals $(F : X \to \mathbb{R})$

- Idea to solve $f(x) = 0$. If we knew that $f = F'$, then the critical points of $F$ would be a solution to $f(x) = 0$.

- Theorem of Mountain Pass Type

**Rolle's Theorem.** Let $f \in C^1([x_1, x_2], \mathbb{R})$. If $f(x_1) = f(x_2)$, then there exists $x_3 \in (x_1, x_2)$ such that $f'(x_3) = 0$. 
Mountain Pass Geometry
**Mountain Pass Theorem**

**Theorem** (Finite Dimensional MPT, Courant). *Suppose that a continuous function* $F : \mathbb{R}^n \rightarrow \mathbb{R}$ *is proper and possesses two distinct strict relative minima* $x_1$ *and* $x_2$. *Then* $F$ *possesses a third critical point* $x_3$ *distinct from* $x_1$ *and* $x_2$, *characterized by*

$$F(x_3) = \inf_{\Sigma \in \Gamma} \max_{x \in \Sigma} F(x)$$

*where* $\Gamma = \{\Sigma \subset \mathbb{R}^n; \Sigma \text{ is compact and connected and } x_1, x_2 \in \Sigma\}$. *Moreover,* $x_3$ *is not a relative minimizer, that is, in every neighborhood of* $x_3$ *there exists a point* $x$ *such that* $F(x) < F(x_3)$. 
Application of MPT

**Hadamard’s Theorem.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism so that $\|f(x)\| \to \infty$ as $\|x\| \to \infty$, then $f$ is a diffeomorphism.

Sketch of Proof: (show that $f$ is onto and injective)

- Check that $f$ is onto.

- Injectivity by contradiction.
  Suppose $f(x_1) = f(x_2) = y$, then define $F(x) = \frac{1}{2} \|f(x) - y\|^2$

  Check the MPT geometry $\Rightarrow \exists x_3, F(x_3) > 0$ (i.e., $\|f(x_3) - y\| > 0$), but

  $$F'(x_3) = \nabla^T f(x_3) \cdot (f(x_3) - y) = 0$$