Foliations and Global Inversion

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The Global Inversion Problem

Program: Understand injectivity mechanism for a local diffeomorphism $f : M \longrightarrow N$ to be invertible. (M and N are non-compact manifolds)

Focus: Use Geometric and Topological methods to understand global invertibility of maps on $\mathbb{R}^n$.

Outline:

I) Motivation

II) Classical results

III) Topological Results

IV) Recent Progress and Holomorphic Results

The Global Inversion Problem

Motivation:

1. Algebraic Geometry.

Jacobian Conjecture: Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a local polynomial biholomorphism, i.e., $\det(DF(z)) = 1$, then F admits a polynomial inverse.

- It suffices to show injectivity.

- Pinchuck has examples of real polynomial maps with non-zero Jacobian determinant everywhere and not injective

Understanding the structure of $\mathrm{Aut}(\mathbb{C}^n)$, $n > 1$.

The Global Inversion Problem

2. Dynamical Systems.

Markus-Yamabe Conjecture: Let $f : \mathbb{R}^n \to \mathbb{R}^n$be a C^1 map with $Spec(Df) \subseteq \{\mathbb{R}e < 0\}$ and $f(0) = 0$, then 0 is a global attractor of $\dot{x} = f(x)$.

- Solved when $n = 2$ by Fessler, Glutsiuk, Gutierrez, 95.

[Gutierrez] If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 map with $Spec(Df) \cap [0, \infty) = \emptyset$, then f is injective.

[Fernandes, Gutierrez, Rabanal, 04] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map, not necessarily C^1. If there is $\varepsilon > 0$ such that $Spec(Df) \cap [0, \varepsilon) = \emptyset$, then f is injective.

- Consider the structure of codimension one foliations of $\mathbb{R}^2$ and its half Reeb components.

The Global Inversion Problem

3. Applied Mathematics - Financial Mathematics

Results to give global invertibility conditions for large supply-demand systems.

[Gale-Nikaido, 72] Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$. F is a bijection provided there are positive numbers m, n and M, N such that

$$m \le |f_x| \le M$$
$$n \le |f_x g_y - f_y g_x| \le N,$$

4. Non-linear Analysis.

In solving PDEs it is desirable to reformulate the problem in operator form and discuss invertibility, i.e., existence and uniqueness.

- Uniformization Theorem for surfaces M^2 with $\chi(M^2) < 0$.

The Classical Approach

Hadamard's "observation". *A local homeomorphism* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is bijective if and only if it is a proper map.*

Recall, a map is **proper** if the pre-image of every compact set is compact. Analytically, f is proper if and only if

$$\lim_{|x|\to\infty} \|f(x)\| = \infty.$$

Idea: Show that f is a covering map.

Def: f is a **cover map** if and only if f lifts line segments [Plastock].

When $f \in C^1$, this is equivalent to find a **global** solution ($t \in [0,1]$) to the following ODE:

$$\frac{dx}{dt} = [Df(x)]^{-1}(y - y_0), \quad x(0) = x_0$$

Analytical - Result

Theorem A (Hadamard-Plastock, 74). Let $f : X \to Y$ be a Banach space local diffeomorphism. If

$$\inf_{x \in \mathbb{R}^n} \|Df(x)^{-1}\|^{-1} > 0,$$

then f is a **diffeomorphism** of X onto Y.

- The finite dimensional case is due to Hadamard (1906).

Well-known application of the result above in Geometry:

[Hadamard] A simply-connected manifold with non-positive sectional curvature is diffeomorphic to $\mathbb{R}^n$.

- Curvature condition implies that the exponential map is a cover.

Analytical - Result

- Arguments are of great conceptual value, but of little use in applications:

It does **not** predict that $(x,y) \mapsto (x+y^3,y)$ is invertible.

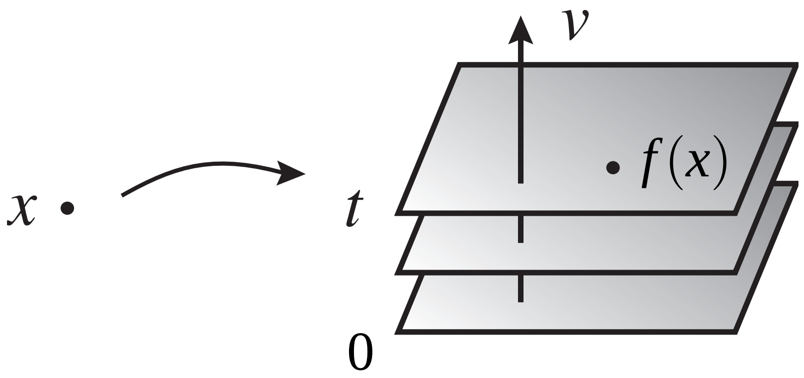
Notice that $Df(x,y) = \begin{pmatrix} 1 & 3y^2 \\ 0 & 1 \end{pmatrix} \rightsquigarrow Df(x,y)^{-1} = \begin{pmatrix} 1 & -3y^2 \\ 0 & 1 \end{pmatrix}$

$$\inf_{(x,y)\in\mathbb{R}^2} \|Df(x,y)^{-1}\|^{-1} = \inf_{(x,y)\in\mathbb{R}^2} \frac{1}{\sqrt{2+9y^4}} = 0.$$

Current Developments

- Substantial improvement to Hadamard-Plastock in **finite** dimensions by Nollet-Xavier using **Degree Theory**.

For $f : \mathbb{R}^n \to \mathbb{R}^n$ and $v \in S^{n-1}$, consider the **height function** $f_v : \mathbb{R}^n \to \mathbb{R}$, $f_v(x) = \langle f(x), v \rangle$.



Theorem B (Nollet-Xavier, 02). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism and g a complete metric on $\mathbb{R}^n$. Then f is bijective if, for all $v \in S^{n-1}$,*

$$\inf_{x \in \mathbb{R}^n} |\nabla^g f_v(x)| > 0.$$

Improvement over Hadamard

Hadamard's condition:

$$\|Df(x)^{-1}\|^{-1} = \|Df(x)^{-1*}\|^{-1} = \|Df(x)^{*-1}\|^{-1}$$
$$= \inf_{|v|=1} |Df(x)^* v| = \inf_{|v|=1} |\nabla f_v(x)|$$

Condition of Theorem A:

$$\inf_{x\in\mathbb{R}^n} \|Df(x)^{-1}\|^{-1} = \inf_{x\in\mathbb{R}^n} \inf_{|v|=1} |\nabla f_v(x)| > 0.$$

Compare to condition in Theorem B:

$$\inf_{x\in\mathbb{R}^n} |\nabla f_v(x)| > 0.$$

This **does** show that $(x,y) \mapsto (x+y^3,y)$ is invertible.

For $v = (v_1,v_2)$, $|\nabla f_v(x,y)| = \sqrt{v_1^2 + (3y^2 v_1 + v_2)^2}$.

$|\nabla f_v(x,y)| > |v_2|$ if $v_1 = 0$ while $|\nabla f_v(x,y)| > |v_1|$ otherwise.

Improvement on Set of Metrics

We generalize Theorem B as follows:

Theorem 1. A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if, for each $v \in S^{n-1}$ there exists a complete Riemannian metric g_v on $\mathbb{R}^n$, such that

$$\inf_{x \in \mathbb{R}^n} |\nabla^{g_v} f_v(x)| > 0 \ .$$

- Note the change in the order of quantifiers. $\left(\forall v \exists g \text{ vs. } \exists g \forall v\right)$

- The result above is a consequence of two topological results developed to detect **Injectivity** and **Surjectivity** using **Intersection Theory** (Linking numbers).

Topological Results

Definition: A space is **acyclic** if it has the **homology** of a point.

Theorem 2. A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is **bijective** if and only if the pre-image of every affine hyperplane is non-empty and acyclic.

Theorem 3. A local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is **injective** if the pre-image of every affine hyperplane is either empty or acyclic.

Analytical Condition and Topology

Topological conditions are natural.

Proposition. If $\inf_{x\in\mathbb{R}^n} |\nabla f_v(x)| > 0$, then $f_v^{-1}(c)$ is nonempty and homotopically trivial for any $c \in \mathbb{R}$.

- $f_v^{-1}(c) = f^{-1}(\text{Hyperplane})$
- $\inf_{x\in\mathbb{R}^n} |\nabla f_v(x)| > 0 \Longrightarrow X_v = \dfrac{\nabla f_v}{|\nabla f_v|^2}$ is **complete**.

For $p \in \mathbb{R}^n$, let φ_p be the maximal integral curve of X_v at p.

$$\frac{d}{dt} f_v\big(\varphi_p(t)\big) = \nabla f_v\big(\varphi_p(t)\big) \cdot \varphi_p'(t) = \big\langle \nabla f_v\big(\varphi_p(t)\big), X_v\big(\varphi_p(t)\big)\big\rangle = 1$$

By integration between $t = 0$ and $t = \tau$

$$f_v\big(\varphi_p(\tau)\big) - f_v\big(\varphi_p(0)\big) = \tau \Rightarrow f_v\big(\varphi_p(\tau)\big) = \tau + f_v(p)$$

Thus $f_v^{-1}(c)$ is non-empty.

Analytical Condition and Topology

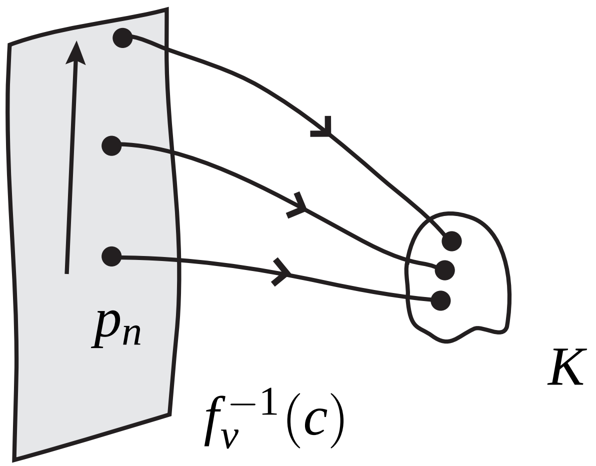
Define $f_v^{-1}(c) \times \mathbb{R} \xrightarrow{\varphi} \mathbb{R}^n$, by $\varphi(p,t) = \varphi_p(t)$.

Claim: φ is a diffeomorphism.

- φ is a local diffeomorphism by transversality of ∇f_v.
- φ is **proper**. $\left(\lim_{|(p,t)| \to \infty} |\varphi(p,t)| = \infty \right)$.

If $|t_n| \to \infty$, done. ($|\varphi(p_n, t_n)| \geq |t_n + c| \to \infty$). Else, if $|t_n| \leq \tau$;

Consider $|\varphi_{p_n}(t_n) - p_n| \leq \int_0^\tau |\varphi'_{p_n}(t)| dt = \int_0^\tau \frac{1}{|\nabla f_v|} dt \leq M$.



Motivation

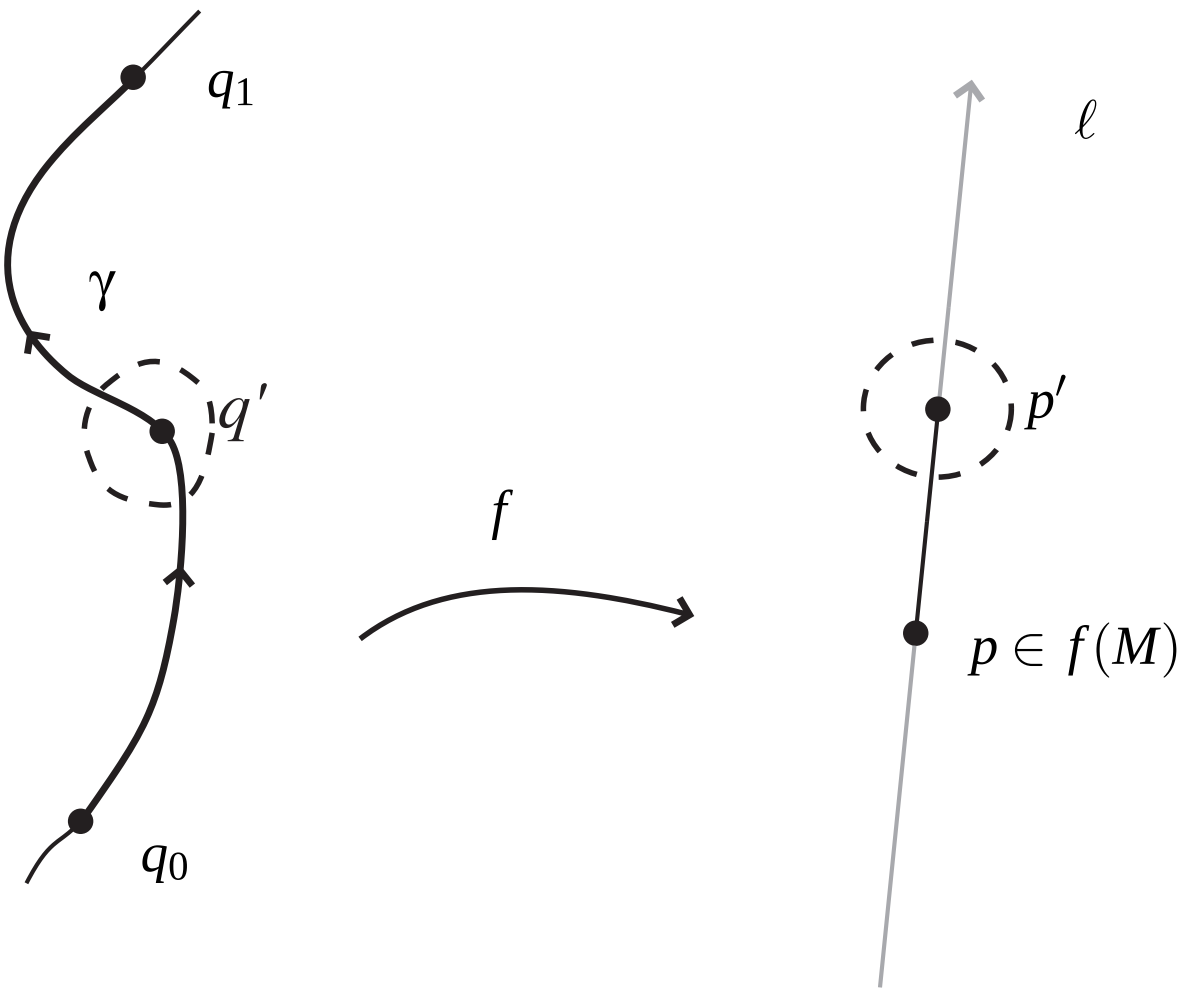
Injectivity is closely related to the topology of the pullback of hyperplanes.

Consider the following near tautology:

"*A local diffeomorphism is injective if and only if the pre-images of every point is connected or empty.*"

Observation. A local diffeomorphism is injective provided that the pre-image of every line is connected or empty.

Motivation



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Q: Can one infer global injectivity from topological information of the pre-images of **higher dimensional** manifolds?

Nollet-Xavier Conjecture. A local diffeomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is injective if the pre-images of every affine hyperplane is **connected** or empty.

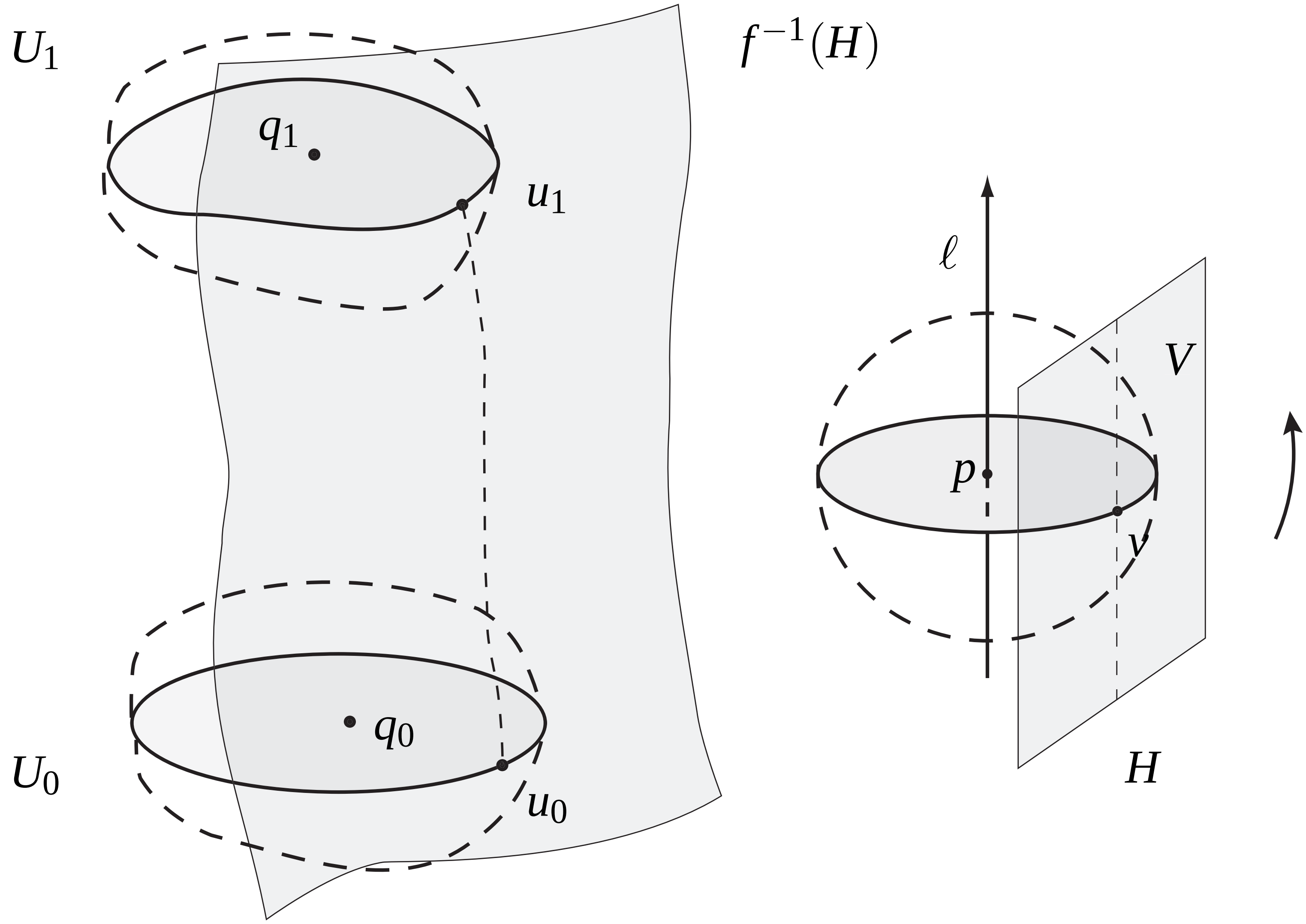
Motivation

Importance: If the conjecture above is true, then the Jacobian Conjecture follows.

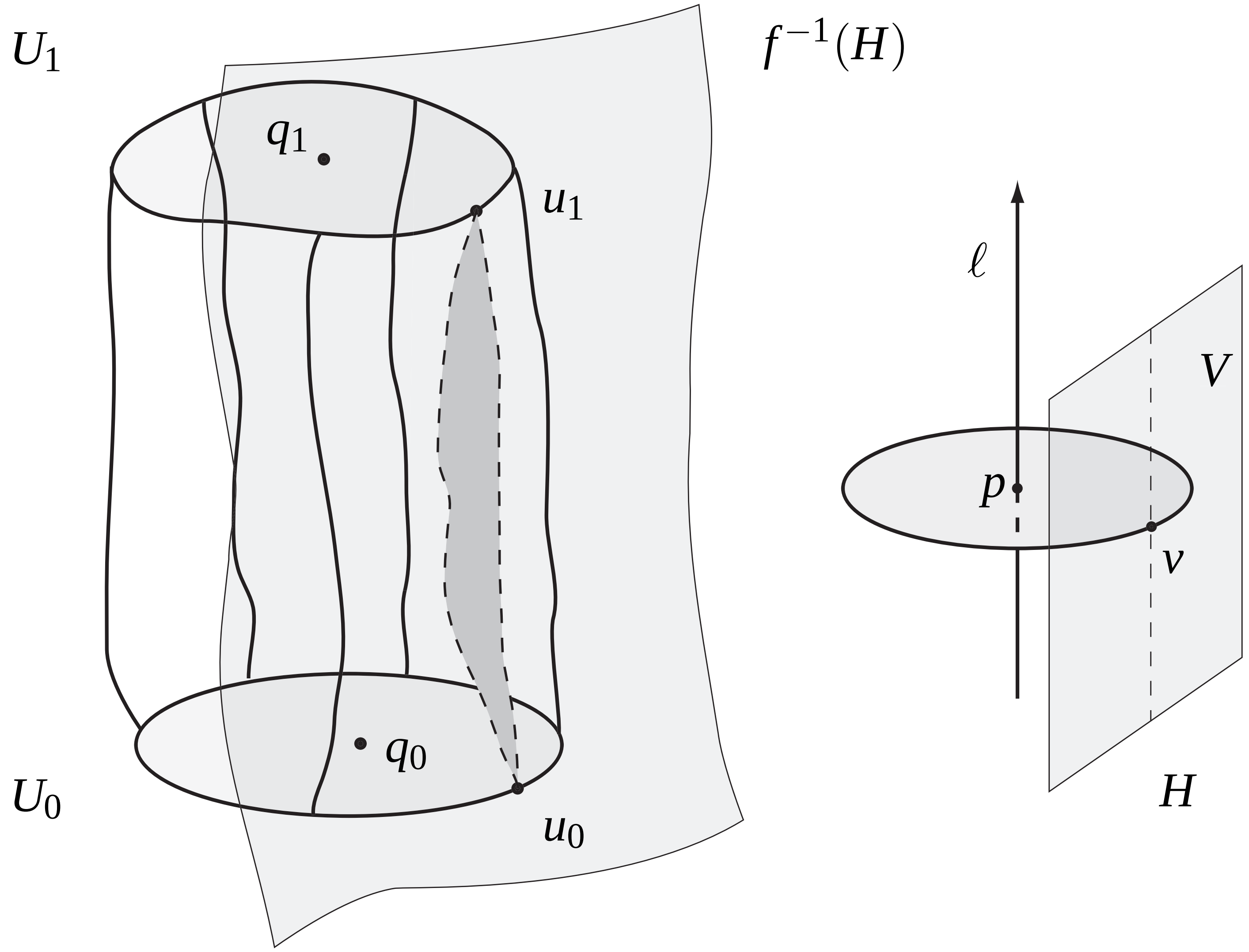
- $F : \mathbb{C}^n \to \mathbb{C}^n$ a polynomial local biholomorphism and $H \subseteq \mathbb{C}^n$ a real hypersurface foliated by complex hyperplanes V
- A Bertini type Theorem [Schinzel] implies that $F^{-1}(V)$ is generically connected.
- Thus $F^{-1}(H)$ is connected.

Proposition. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a local diffeomorphism so that the pre-image of every affine plane is connected and simply-connected, i.e., a topological plane, then f is injective.

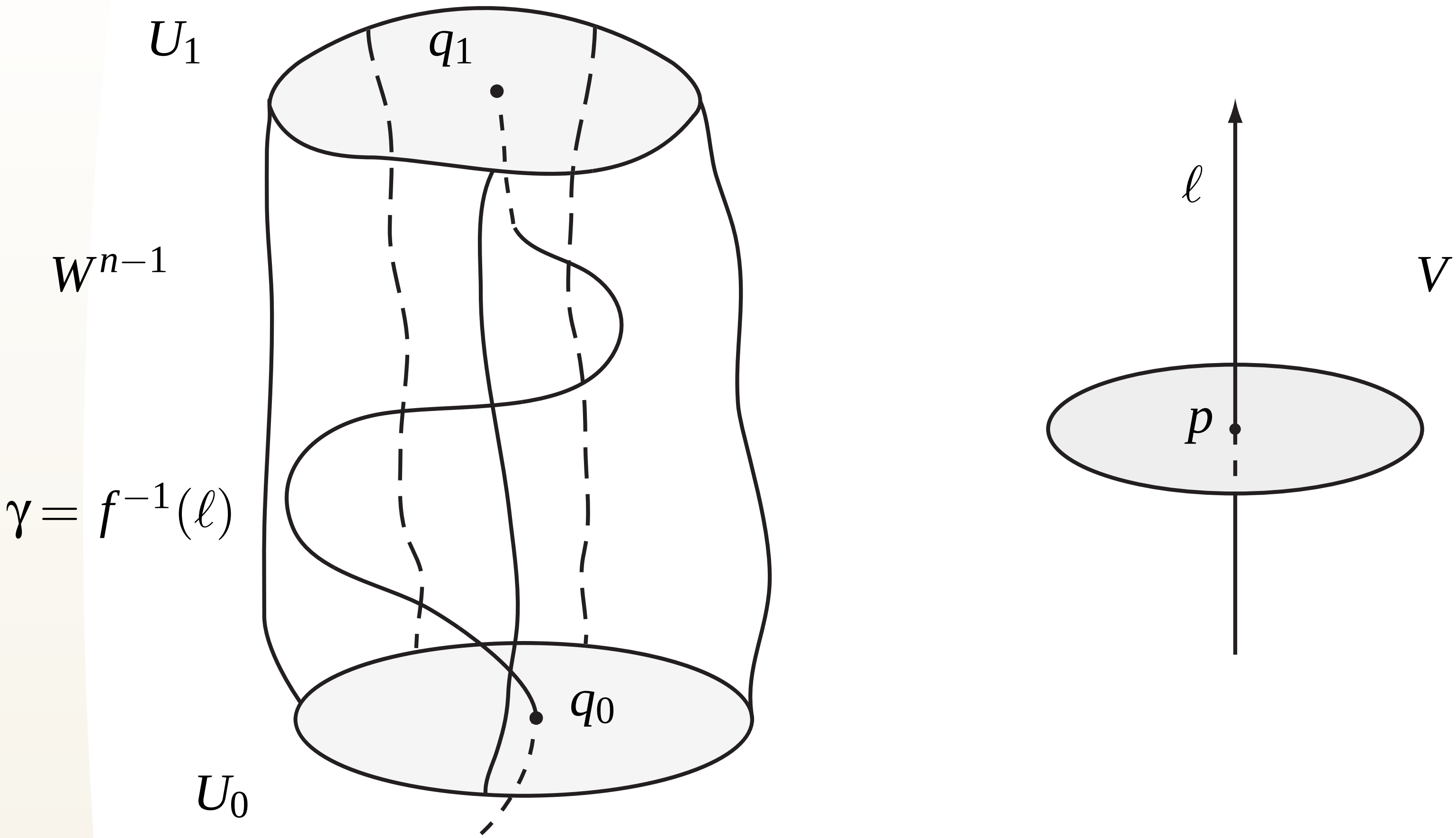
Outline of Proof - Injectivity



Outline of Proof - Injectivity



Computation of Intersection Number



$$0 = \#(\Gamma^{n-1},\gamma) = \#(U_0,\gamma) + \#(W^{n-1},\gamma) + \#(U_1,\gamma)$$

$$0 = \#(\Gamma^{n-1},\gamma) = \quad 1 \qquad +0 \qquad\qquad +\#(U_1,\gamma)$$

General Injectivity Result

Theorem 4. For $n \geq 3$, let M be a connected smooth manifold with $H_{n-1}(M) = 0$ and $f : M \to \mathbb{R}^n$ be a local diffeomorphism. If there is a line ℓ such that the pre-image of every affine hyperplane parallel to ℓ is either empty or acyclic, then f is injective.

Idea of proof (by contradiction)

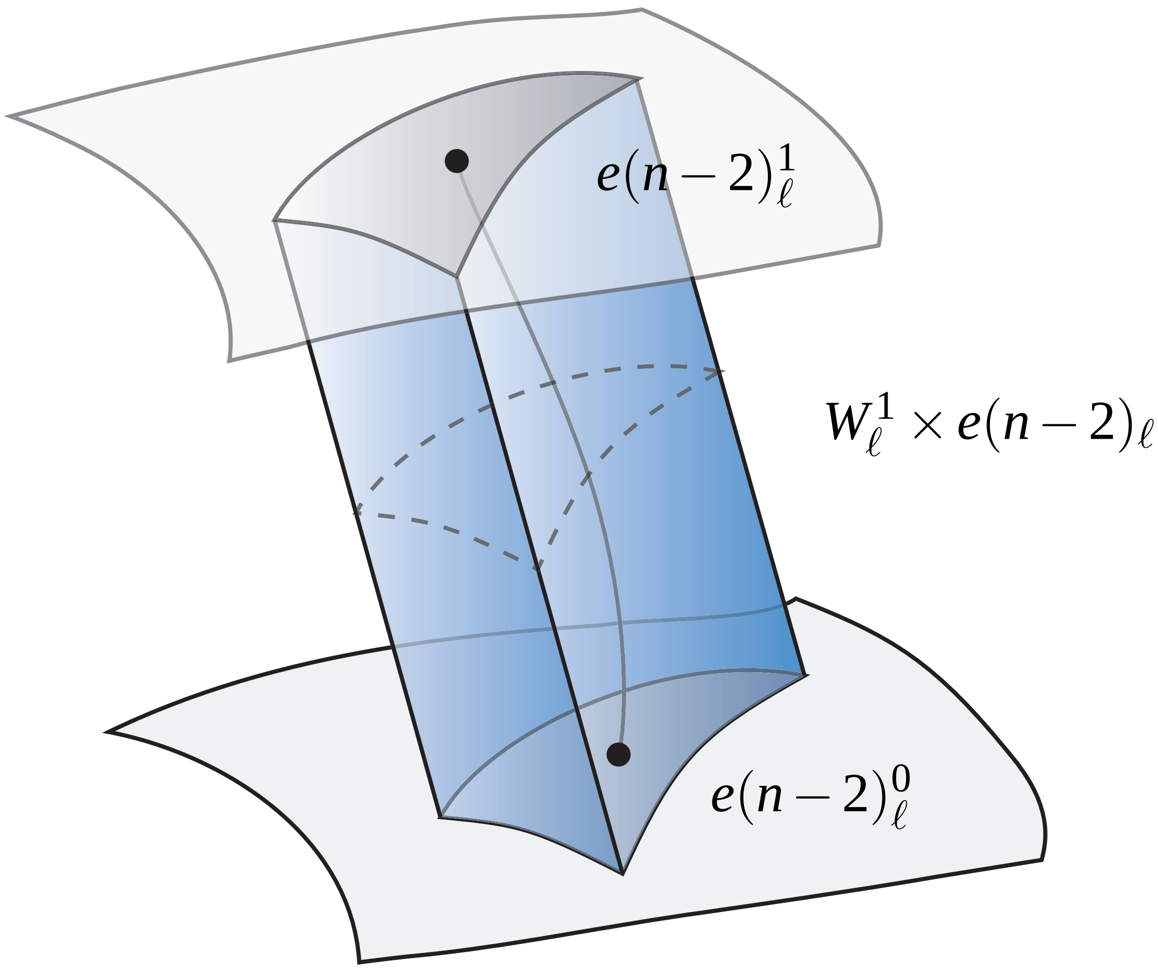
- Construct a closed (geometric) chain complex Γ^{n-1} [Topological Cylinder]

- Combinatorial topology to keep track of intersections [Triangulation of S^{n-2}].

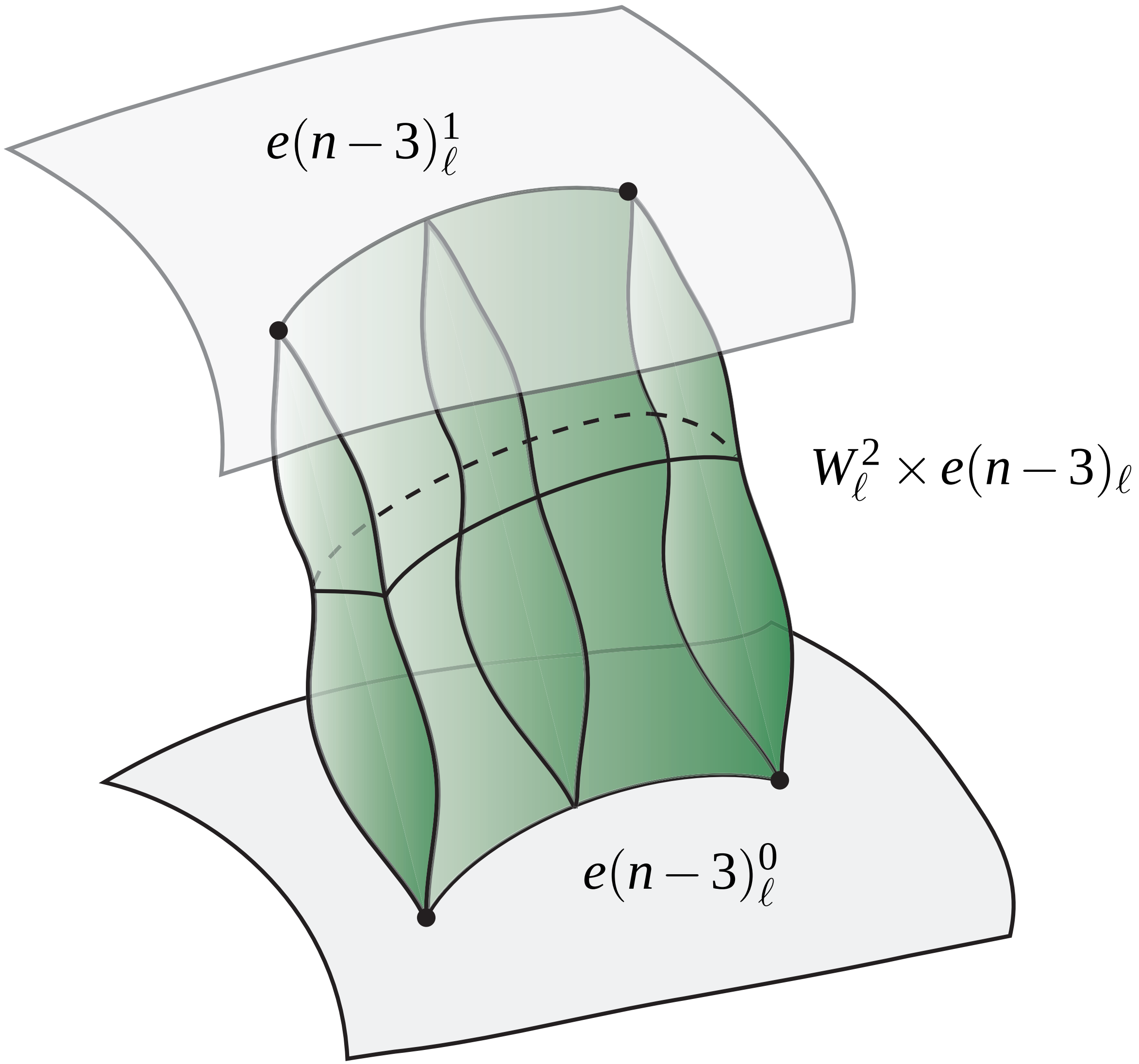
- Inductive process from top dimensional cells to vertices.

- Compute the intersection number in domain M, the hypotheses of $H_{n-1}(M) = 0$ implies that $\#(\Gamma^{n-1}, \gamma) = 0$.

Cycle Construction - Step 0



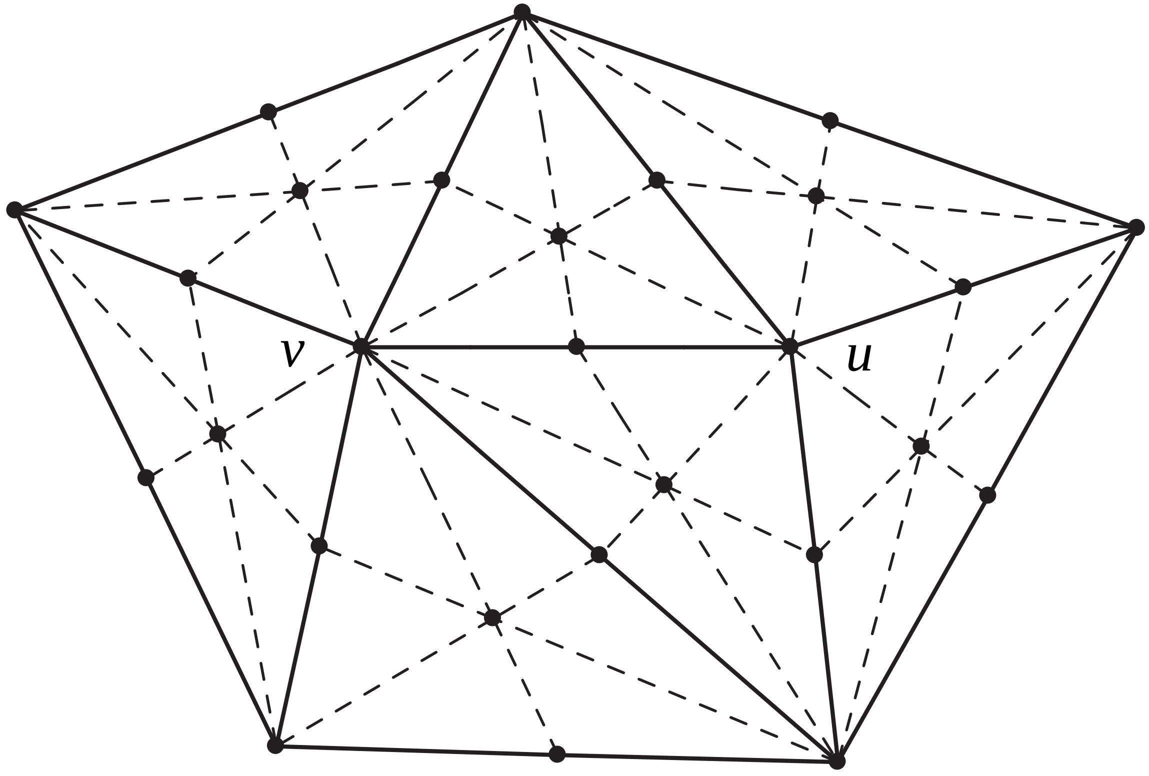
Cycle Construction - Step 1



Cycle Construction - General Step k

- Given $\sigma = e(n-2-k)_\ell$ a cell of S^{n-2}, we must find all $(n-1-k)$-cells that contain σ.

- Take all $(k-1)$-cells in $\partial e^{\vee}(k)_\ell$ (or the vertices in $\mathrm{Link}(\sigma, S^{n-2})$).



Cycle Construction

- Define

$$W^{n-1} = \sum_{k=0}^{n-2} \overbrace{\sum_{\ell \in E(n-2-k)} W_\ell^{k+1} \times e(n-2-k)_\ell}^{S_k}.$$

$$\partial S_0 = (\partial X_0 + \partial X_1) \quad + \sum_{\ell \in E(n-2)} \sum_{j_{n-3}} W_\ell^1 \times e(n-3)_{j_{n-3}}$$

$$\partial S_1 = \sum_{\ell \in E(n-3)} \left[\sum_{j_{n-2}} W_{j_{n-2}}^1 \times e(n-3)_\ell + \sum_{j_{n-4}} W_\ell^2 \times e(n-4)_{j_{n-4}} \right]$$

$$\partial S_k = \sum_{\ell \in E(n-2-k)} \left[\sum_{j_{n-1-k}} W_{j_{n-1-k}}^k \times e(n-2-k)_\ell + \sum_{j_{n-3-k}} W_\ell^{k+1} \times e(n-3-k)_{j_{n-3-k}} \right]$$

$$\partial S_{n-2} = \sum_{\ell \in E(0)} \sum_{j_1} W_{j_1}^{n-2} \times e(0)_\ell \quad + \quad 0$$

General Surjectivity Result

Theorem 5. Let M be a connected smooth manifold and $f : M \longrightarrow \mathbb{R}^n$ be a local diffeomorphism. If the pre-image of every affine hyperplane is non-empty and acyclic, then f is surjective.

- Argument with intersection numbers (Linking numbers) is done in $\mathbb{R}^n$, hence no need of homological hypotheses on domain M.

- If one allows the pre-image of one hyperplane to be non-empty, f is not surjective, $(x,y) \mapsto (y - e^x, y + e^x)$.

Outline of Surjectivity

Expand the image around a point, say $f(o) = 0 \in f(\mathbb{R}^n)$, until $B(0,R) \subset f(\mathbb{R}^n)$ for some $R > 0$.

Trivial example: $f = Id : \mathbb{R}^n \to \mathbb{R}^n$.

For $v \in S^{n-1}$, let $\mathcal{H}_v$ be the canonical hyperplane foliation of $\mathbb{R}^n$ by parallel hyperplanes orthogonal to v.

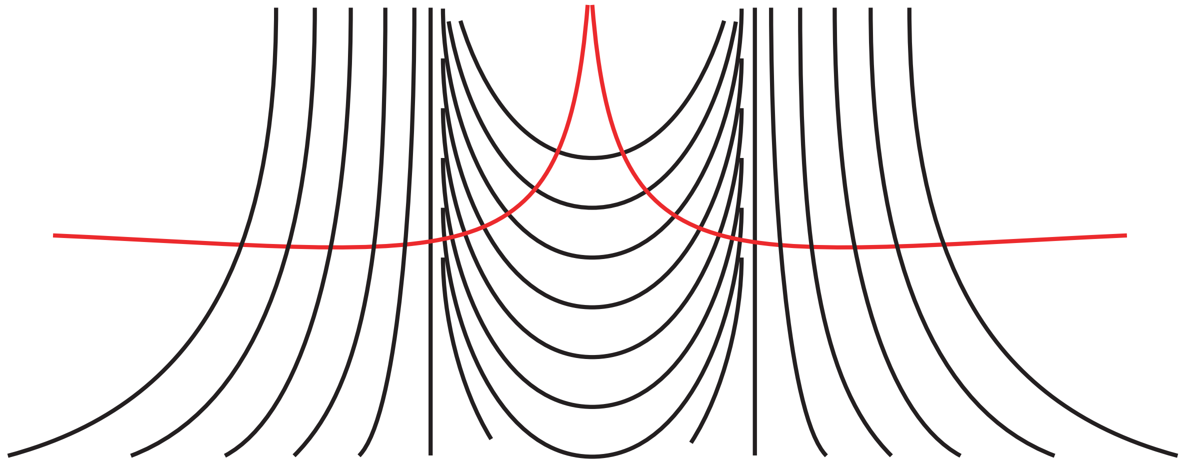
Extend $B(0,\varepsilon)$ **in domain** along radial paths γ_v which are **global transversals** to the pullback foliation $f^*\mathcal{H}_v$.

Global Transversal

The **non-emptiness** and **connectedness** of pre-images of hyperplanes ensures the existence of a global transversal γ_v to $f^*\mathcal{H}_v$.

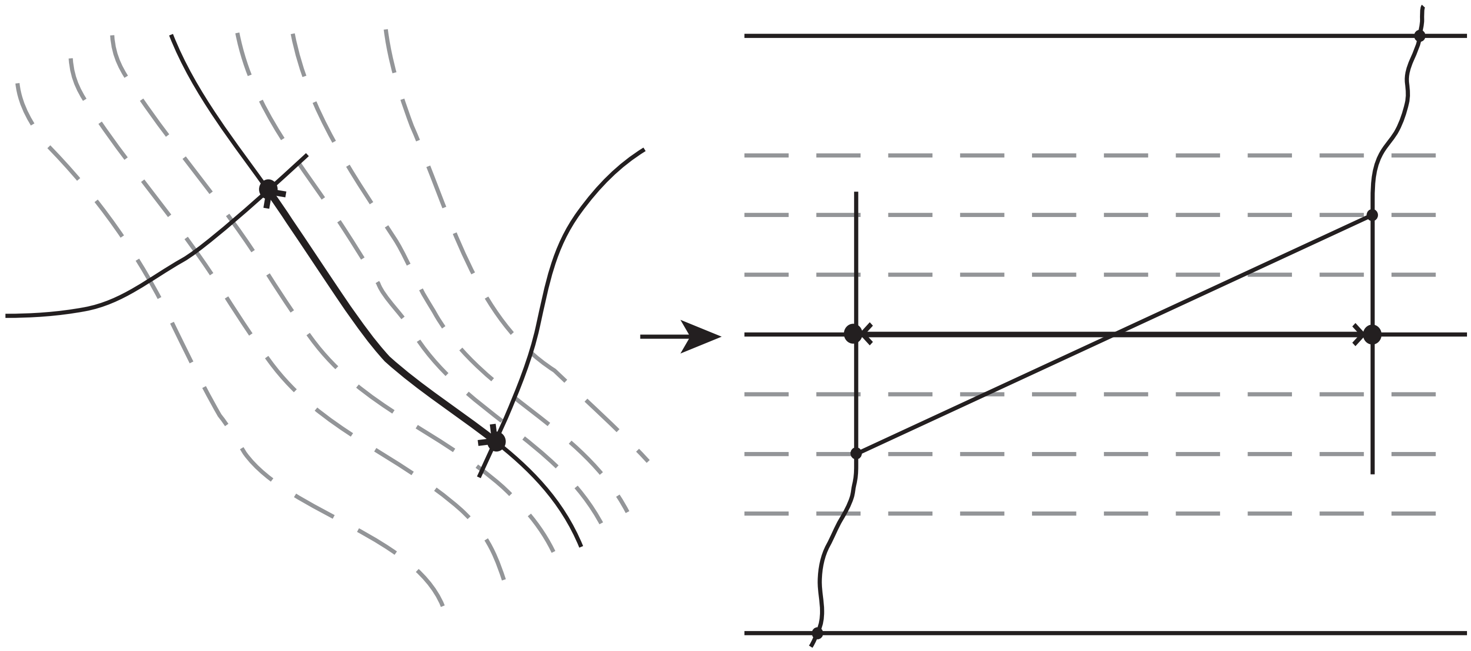
This is not true in general for a codimension 1 foliation.

- $f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = (x^2 - 1)e^y.$



Global Transversal

In general, γ_ν is constructed using Transverse Modification techniques and the **Global Trivialization Lemma**.



Idea of Surjectivity

- Construct a family of geometric chain complexes $W^{n-1}(t)$ that represent homologous cycles in $M \setminus \{o\}$ relative to t, for $t \in [\varepsilon, R]$.

  (Intuitively, a family of expanding spheres)

- Computation of Linking number between $p \in \partial B(0,R)$ and $f\left(W^{n-1}(t)\right)$.

A bit of notation:

- Let $N_v(t) = f^{-1}(\mathcal{H}_v(t))$ (leaves of pullback foliation).
- $W = W^{n-1}(\varepsilon) = f^{-1}(\partial B(0,\varepsilon))$ (component near o).

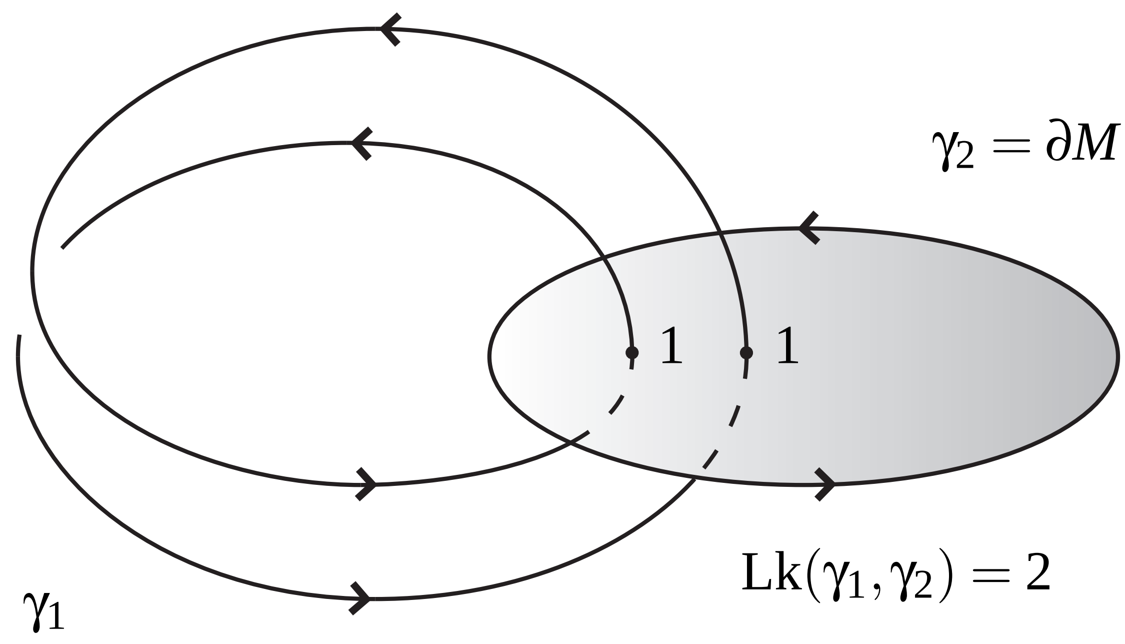
Linking Numbers

Let X^p and Y^{q-1} be two nonintersecting **cycles** in $\mathbb{R}^n$ with $p+q=n$. Suppose $X^p = \partial Z^{p+1}$. The **linking number** $\mathrm{Lk}(X^p, Y^{q-1})$ is given by:

$$\mathrm{Lk}(X^p, Y^{q-1}) = \#(Z^{p+1}, Y^{q-1})$$

Fact: This is independent of the choice of the bounding chain.

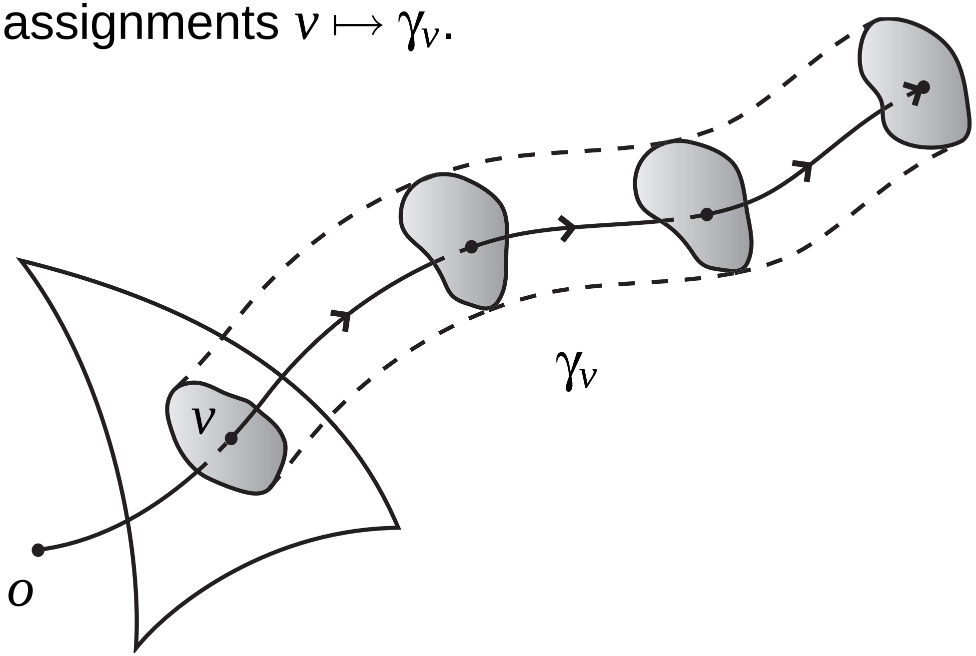
Example:



Cycle Construction - Initial Step

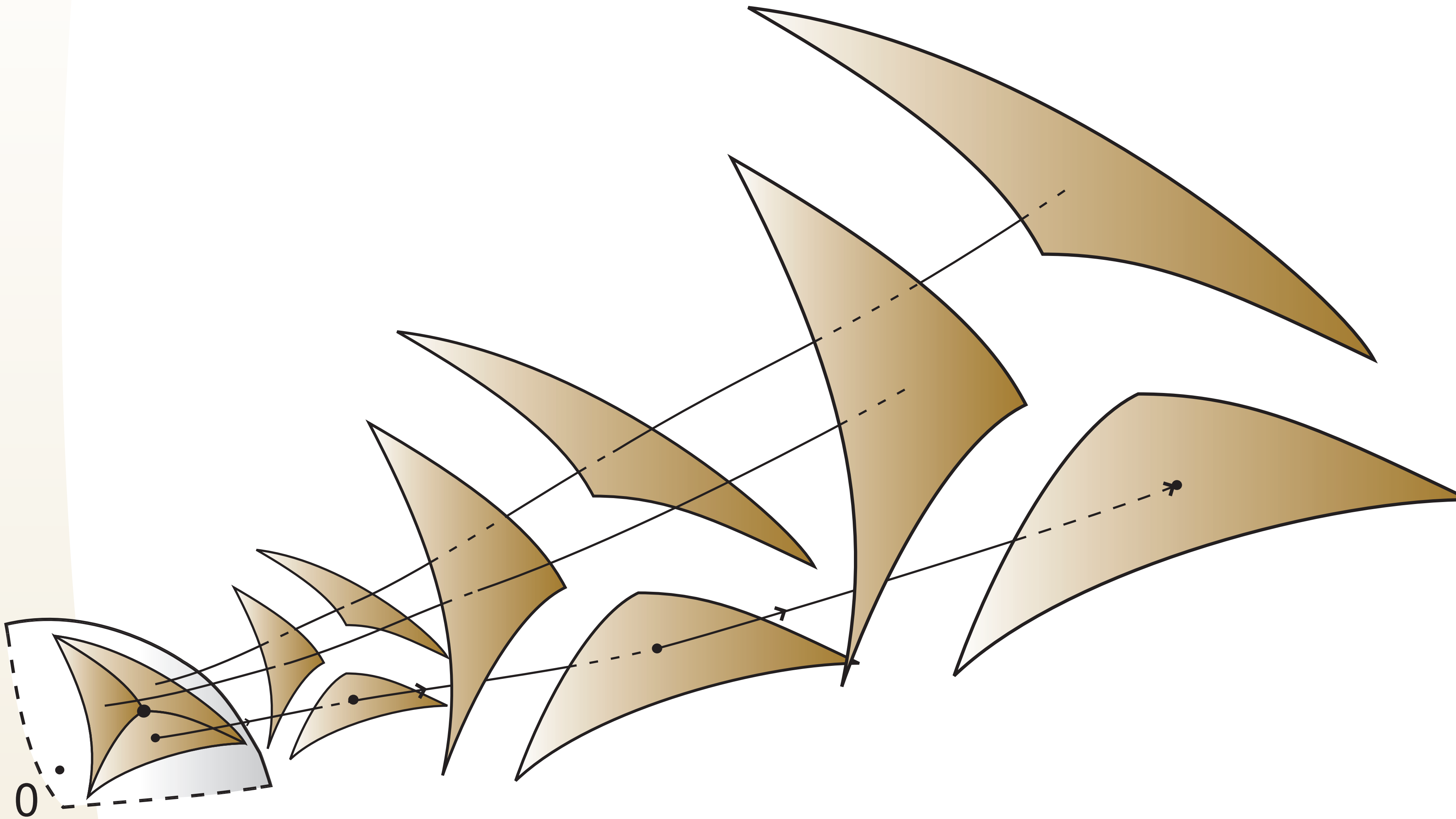
Triangulate $\partial B(0,\varepsilon)$ and consider the induced triangulation on W.

From connectedness (0-homology), it is possible to find **local** (*continuous*) assignments $v \mapsto \gamma_v$.

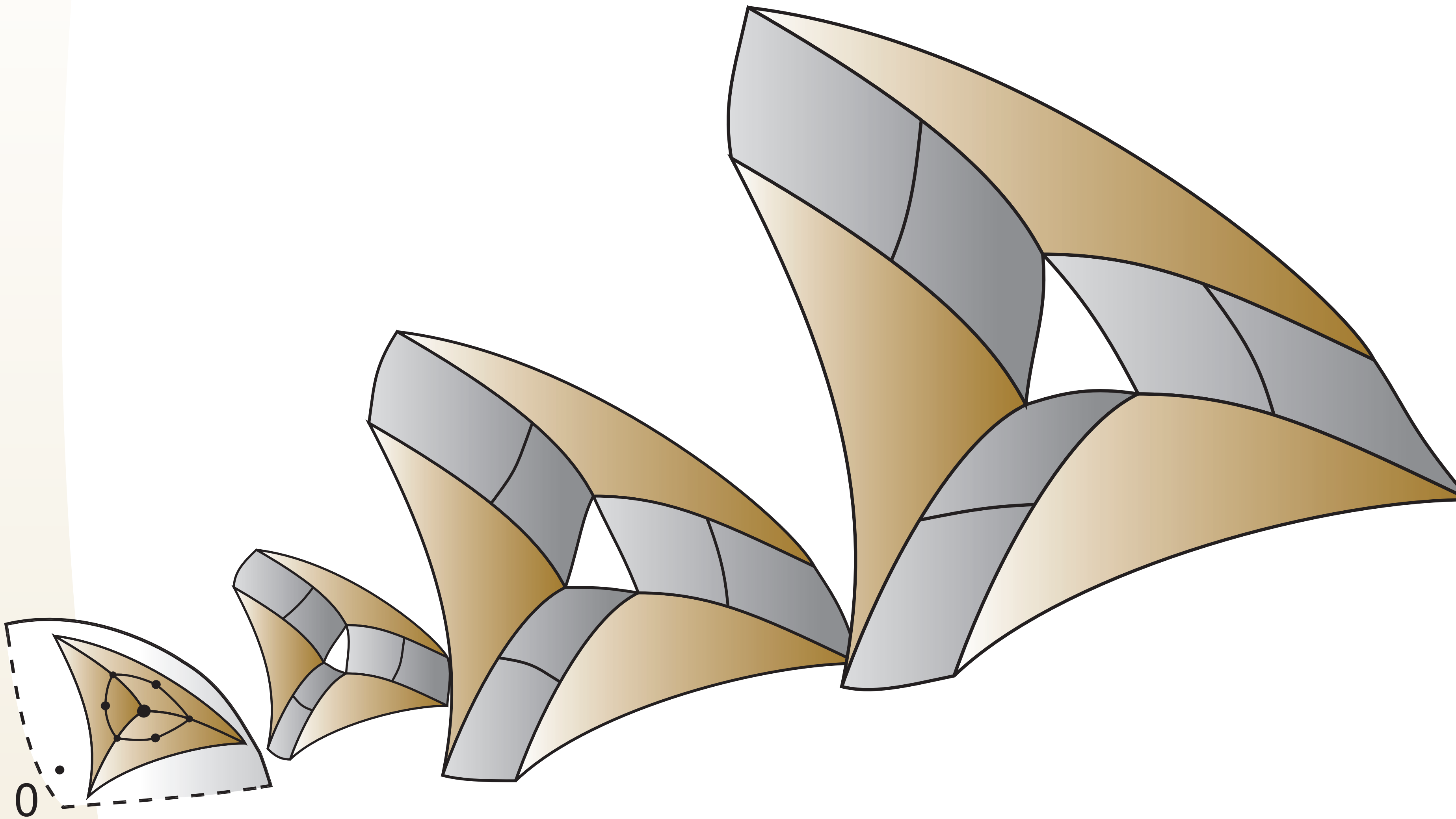


- The main difficulty is a **global assignment** $S^{n-1} \ni v \mapsto \gamma_v$.
- Locally, obtain 1-parameter family of $(n-1)$-chains

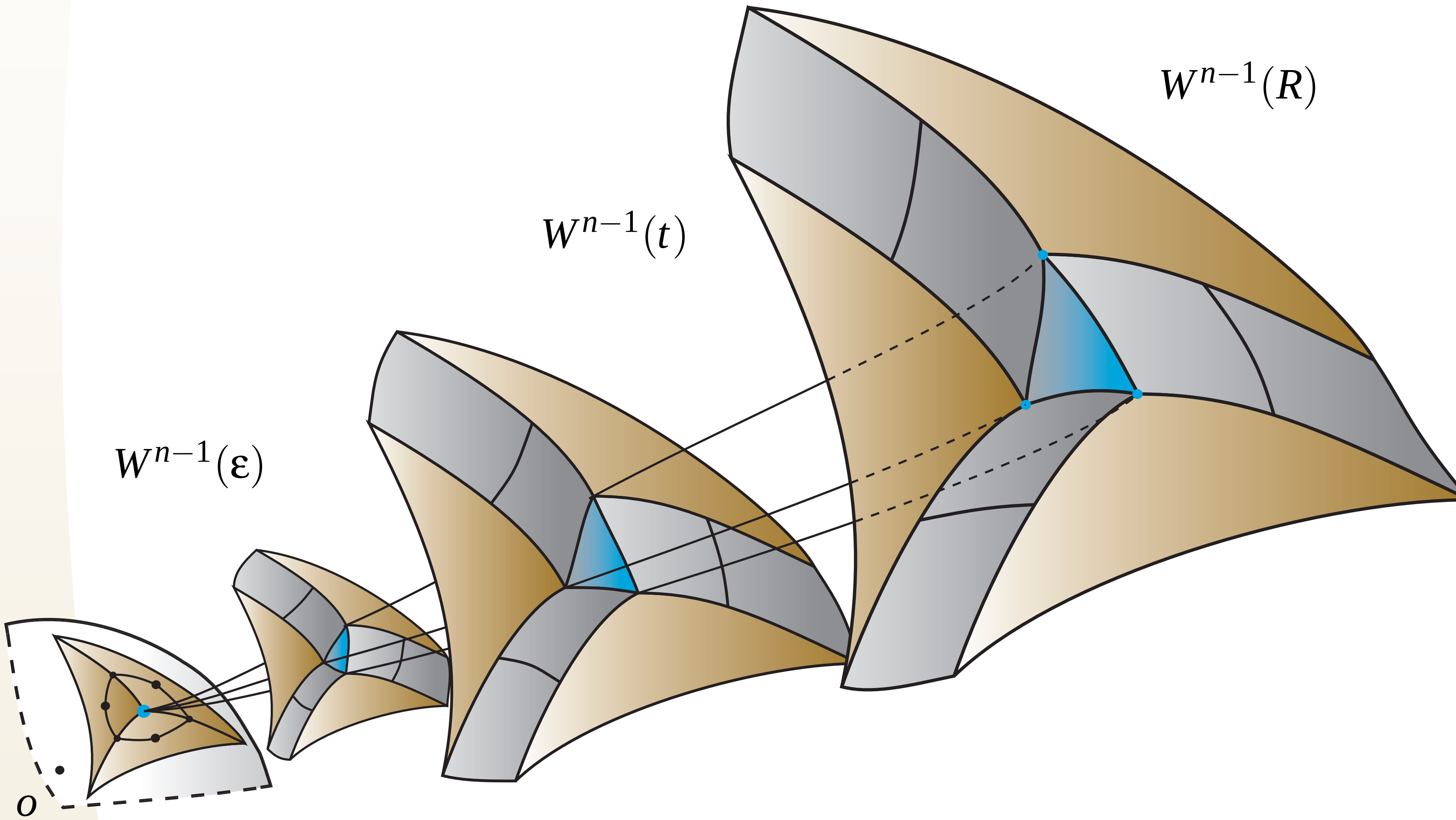
Cycle Construction - Step 0



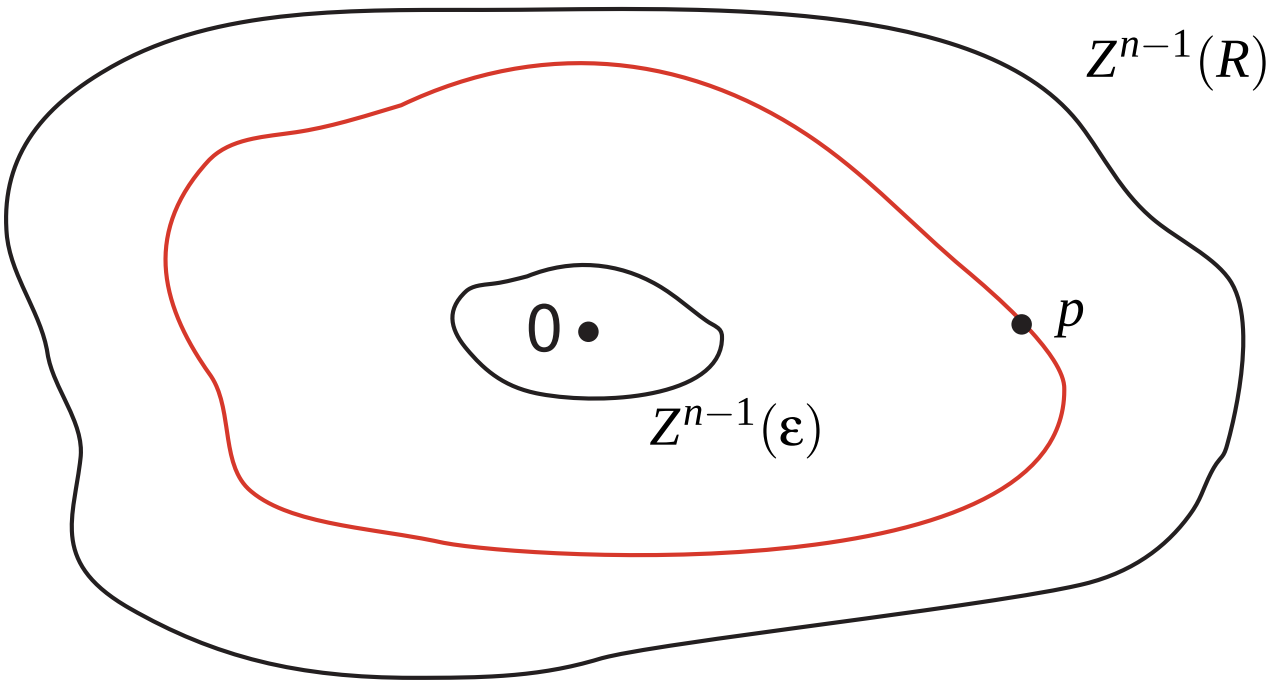
Cycle Construction - Step 1



Cycle Construction - Step k



Surjectivity argument



Let $f\left(W^{n-1}(t)\right) = Z^{n-1}(t)$.

For $p \in \partial B(0,R)$, compute $\mathrm{Lk}(p, Z^{n-1}(\varepsilon))$ and $\mathrm{Lk}(p, Z^{n-1}(R))$.

$\mathrm{Lk}(p, Z^{n-1}(R)) = \mathrm{Lk}(0, Z^{n-1}(R)) = \mathrm{Lk}(0, Z^{n-1}(\varepsilon))$.

- $Z^{n-1}(t) \sim Z^{n-1}(s)$ in $f(\mathbb{R}^n) \setminus \{0\}$.

Thus, $p \in Z^{n-1}(t)$ for some t.

Recent Progress

- It is desirable to find Analytic Conditions to detect Topological Information.

- How simple does the Topology need to be?

- Use the structure of Algebraic Geometry to global injectivity of local biholomorphism.

Recent Progress - Analytic Results

Theorem C (Sabatini, 93). Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local diffeomorphism. If

$$\int_0^\infty \inf_{\|x\|=r} \|\nabla f_1(x)\| \, dr = \infty,$$

then f_1 assumes every real value and f is injective.

Theorem D (Sabatini, 93). Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a local diffeomorphism. If

$$\int_0^\infty \inf_{\|x\|=r} \frac{|\det Df(x)|}{\|\nabla f_2(x)\|} \, dr = \int_0^\infty \inf_{\|x\|=r} \frac{|\det Df(x)|}{\|\nabla f_1(x)\|} \, dr = \infty,$$

then f is a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$.

- Although methods are two dimensional in nature, we extend this to all dimensions.

Recent Progress - Analytic Results

Since $\|\nabla f_1(x)\| \geq \dfrac{|\det Df(x)|}{\|\nabla f_2(x)\|}$, Theorem D implies Theorem C.

Notation:

$$|\det Df(x)| = \| \wedge_{1\leq j\leq n} \nabla f_j(x)\| = |\det(\langle \nabla f_j(x), \nabla f_k(x)\rangle)_{1\leq j,k\leq n}|^{1/2},$$

$$\| \wedge_{\substack{1\leq j\leq n\\ j\neq i}} \nabla f_j(x)\| = |\det(\langle \nabla f_j(x), \nabla f_k(x)\rangle)_{\substack{1\leq j,k\leq n\\ j,k\neq i}}|^{1/2}.$$

Recent Progress - Analytic Results

Conditions of Sabatini actually detect topological conditions on the pre-images of affine subspaces.

Theorem 6. For $n \geq 2$, let $f = (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a local diffeomorphism, k an integer with $1 \leq k \leq n$, and H an affine subspace of codimension k. Assume that

$$\int_0^\infty \inf_{\|x\|=r} \frac{\| \bigwedge_{1 \leq j \leq n} \nabla f_j(x) \|}{\| \bigwedge_{\substack{1 \leq j \leq n \\ j \neq i}} \nabla f_j(x) \|} \, dr = \infty \text{ , for each } i = 1, \ldots, k.$$

Then $f^{-1}(H)$ is non-empty and $(n-k)$-connected.

Recent Progress - Analytic Results

Corollary 1. For $n \geq 2$, let $f = (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism such that, for each $i = 1, \ldots, n-1$, we have

$$\int_0^\infty \inf_{\|x\|=r} \frac{\| \bigwedge_{1 \leq j \leq n} \nabla f_j(x) \|}{\| \bigwedge_{\substack{1 \leq j \leq n \\ j \neq i}} \nabla f_j(x) \|} \, dr = \infty \ ,$$

then f is injective.

Corollary 2. For $n \geq 2$ a local diffeomorphism $f = (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective provided that, for all $i = 1, \ldots, n$,

$$\inf_{x \in \mathbb{R}^n} \frac{|\det Df(x)|}{\prod_{1 \leq j \leq n, j \neq i} \|\nabla f_j(x)\|} > 0 \ .$$

- One can view condition in Cor. 2 as an incompressibility condition of the faces of $Df(x)([0,1]^n)$.

Recent Progress - Topological Results

Ideas from Foliation Theory

Theorem 6 (Andrade, ____, 08). Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a local diffeomorphism. If the pre-image of every affine plane is either empty or connected and diffeomorphic to a finitely punctured plane, then f is injective.

- Iteration argument of the Cylinder construction.

- Careful choice of paths at each vertex of triangulation.

Recent Progress - Topological Results

General Result:

Theorem 7 (Andrade, _____, 08). Let $f : M \to \mathbb{R}^3$ be a local diffeomorphism where M is a connected smooth manifold with $H_2(M) = 0$. If there is a line $\ell \subseteq \mathbb{R}^3$ such that the pre-image of every affine hyperplane H parallel to ℓ is either empty or connected with $H_1(f^{-1}(M))$ finitely generated, then f is injective.

Recent Progress - Holomorphic Results

Theorem E (Nollet, Xavier, 04). For $n \geq 2$, let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a local biholomorphism. If $f^{-1}(\ell)$ is 1-connected for every complex line ℓ, then f is injective.

Idea: Assume $f(a) = f(b) = 0$.

$f^{-1}(\ell)$ = Hadamard surface $\Longrightarrow \exists!$ geodesic joining a to b.

$$\mathbb{C}P^{n-1} \xrightarrow{s} S^{2n-1} \cap \ell \ \text{ given by } \ell \mapsto \frac{df(a)v_\ell}{|df(a)v_\ell|}$$

s = **continuous** restriction of the Hopf map $S^{2n-1} \xrightarrow{\pi} \mathbb{C}P^{n-1}$, a contradiction.

$$\begin{tikzcd} H^2(\mathbb{C}P^{n-1}) \arrow[r, "\pi^*"] \arrow[rr, bend left] & H^2(S^{2n-1}) \arrow[r, "s^*"] & H^2(\mathbb{C}P^{n-1}) \end{tikzcd}$$

Recent Progress - Holomorphic Results

Theorem 8 (_____, Xavier, 08). Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial local biholomorphism, then the pre-image of **every** complex line is connected.

- From Algebraic and Complex structure, generically always true.

- View $\mathbb{C}^2 \approx \mathbb{R}^4$.

- Implement geometric construction and intersection number argument for 2-dim complexes.

Recent Progress - Analytic Results

Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 map.

The level surfaces $\{f_i = \text{constant}\}$ make up a codimension one C^2-foliation $\mathcal{F}_i$ on $\mathbb{R}^3$.

Theorem G (Gutierrez, Maquera, 06). If, for some $\varepsilon > 0$, $Spec(Df) \cap (-\varepsilon, \varepsilon) = \emptyset$, then $\mathcal{F}_i$ is a foliation by planes.

- Consequently, there is a foliation $\mathcal{F}$ in $\mathbb{R}^2$ such that $\mathcal{F}_i$ is conjugate to the product of $\mathcal{F}$ by $\mathbb{R}$.