Review Problems for Exam 1

Exam 1 will cover sections 1.2, 1.3, 1.4, 1.5, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4. You should be able to state the definition of major concepts, state named results including idea of proofs, as well as verify if statements are true or false,. There will be six other problems in the exam. Four of them will come out of this review and two problems will be new but based on homework problems.

Hint: These hints are not the actual solution nor contain the entire solution, they are simply given to provide an idea or direction to the actual solution.

Problem 1. Let $x \in \mathbb{R}$ be such that $x + \dfrac{1}{x}$ is an integer. Show that $x^n + \dfrac{1}{x^n}$ is an integer for all $n \in \mathbb{N}$.

Hint: Use the Principle of Mathematical Induction and the binomial formula for $\left(x + \dfrac{1}{x}\right)^n$.

Problem 2. Show that $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \leq \dfrac{1}{\sqrt{2n+1}}$. Use this to compute $\lim\limits_{n \to \infty} \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

Hint: Use the Principle of Mathematical Induction.

Problem 3. Let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \cdots \cup \mathbb{N}_k$ and assume that $\lim\limits_{n \in \mathbb{N}_1} x_n = \lim\limits_{n \in \mathbb{N}_2} x_n = \cdots = \lim\limits_{n \in \mathbb{N}_k} x_n = \ell$. Show that the sequence $\{x_n\}$ converges to ℓ. Does this statement remain true if we allow an infinite decomposition of $\mathbb{N}$, that is, for a decomposition $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \cdots \cup \mathbb{N}_k \cup \cdots$?

Hint: Clearly finiteness plays a role on this result. Try to use the infinite indexes to force a subsequence to be unbounded while keeping $\lim\limits_{n \in \mathbb{N}_k} x_n = \ell$

Problem 4. For each $n \in \mathbb{N}$, let $0 \leq t_n \leq 1$. Assume $\lim x_n = \lim y_n = \ell$. Show that $\lim\,(t_n x_n + (1 - t_n) y_n) = \ell$.

Hint: Use the triangle inequality and choose a suitable N that forces $|x_n - \ell| < \varepsilon/2$ and $|y_n - \ell| < \varepsilon/2$ for $n \geq N$. Alternatively, you can use the algebraic properties of limits to find the desired quantity, observe that you will need to use the fact that t_n is bounded.

Problem 5. Let $a > 0$ and define $x_n = \sqrt[n]{a}$. Show that $\lim x_n = 1$.

Hint: Prove that x_n is monotone and bounded. Then via a suitable subsequence compute the limit, say $\{x_{n^2+n}\}$.

Problem 6. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers. Show that the following are equivalent (in which case we say φ is a **proper** function on $\mathbb{N}$).

i) $\lim_{n\to\infty} \varphi(n) = +\infty$

ii) For all $k \in \mathbb{N}$, $\varphi^{-1}(k)$ is a finite subset of $\mathbb{N}$.

iii) For all finite subset $F \subset \mathbb{N}$, $\varphi^{-1}(F)$ is finite.

Hint: Use the fact that a subset of $\mathbb{N}$ is finite if and only if it is bounded.

Problem 7. Recall that we can define the natural number e as $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Show that for any $n \in \mathbb{N}$, $e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) < \frac{1}{n \cdot n!}$. Conclude that e is irrational.

Hint: Use a suitable comparison with a geometric series to the remaining terms in $e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$ in order to establish the inequality. For the final and perhaps surprising conclusion, suppose that $e = \frac{p}{q}$ and view the just established inequality in the case $n = q$.

Problem 8. Assume $\sum a_n^2$ converges. Show that $\sum \frac{a_n}{n}$ converges as well.

Hint: Prove the Cauchy-Schwarz inequality.

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Use this to show that if $\sum_{i=1}^{\infty} a_i^2$ converges and $\sum_{i=1}^{\infty} b_i^2$ converges, then $\sum_{i=1}^{\infty} |a_i b_i|$ converges.

Problem 9. Let $\{a_n\}$ be a decreasing sequence of real numbers such that $\sum a_n$ converges. Show $\lim na_n = 0$.

Hint:

Initially, show that $a_n \geq 0$. Next, prove the result by contradiction. Assuming $\lim na_n \neq 0$ the goal goal is to find a contradiction to the fact that the sequence of partial sums follows the Cauchy criterion for convergence.

Show that for a given ε, we can find a subsequence $\{a_{n_k}\}$ with the property that $n_k a_{n_k} \geq \varepsilon$. Then for $\varepsilon' = \varepsilon/2$ argue that $|s_m - s_n| < \varepsilon/2$, for some m, n. As you write out $|s_m - s_n|$ in order to try to obtain a reverse inequality, try to show that if $a_n \leq \frac{\varepsilon}{2n_0}$, for a suitable choice of n_0, one can find m, n large enough so that $|s_m - s_n| \geq \varepsilon/2$

Problem 10. Show that a set F is closed if and only if F^c is open.

Hint: Use the definition

Problem 11. [3.2.13] Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set.

Hint: Use the structure theorem of open sets in $\mathbb{R}$ stated in class.

Problem 12. Show a set $X \subseteq \mathbb{R}$ is bounded if and only if any infinite subset of X has a limit point.

Hint: Use the Bolzano-Weierstrass theorem.

Problem 13. [Page 87 #3.3.8]

Hint: Outlined in textbook.

Problem 14. [3.2.8] Given $A \subseteq \mathbb{R}$, let A' be the set of all limit points of A. Show that A' is closed.

Hint: Use a "diagonal" method to compute the limit of the limits and find a sequence of elements in A that converge to such limit. Recall that in such cases it is standard to pick for each $n \in \mathbb{N}$, $\varepsilon_n = \frac{1}{n}$ and find your a_n.

Problem 15. [3.2.12] Decide whether the following propositions are true or false. Provide counterexamples for those that are false and supply proofs for those that are true.

a) For any set $A \subseteq \mathbb{R}$, $\overline{A}^c$ is open.

b) If a set A has an isolated point, it cannot be an open set.

c) A set A is closed if and only if $\overline{A} = A$

d) If A is a bounded set, then $s = \sup A$ is a limit point of A.

e) Every finite set is closed.

f) An open set that contains every rational number must necessarily be all of R.

Hint: Use definitions.

Problem 16. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, show that $A = \{m + n\alpha | m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, that is, given $x \in \mathbb{R}$ and $\varepsilon > 0$ we have $V_\varepsilon(x) \cap A \neq \emptyset$.

Hint: This problem and solution were outlined in the first talk on the Major's seminar. Consider the problem on a circle, that is, argue that one only needs to show that A is dense in $[0, 1]$. Show that there cannot be periodic orbits. So you must have infinitely many points in $[0, 1]$. Then use the additive properties of A to translate your work throughout $[0, 1]$, that is, once you are able to find one convergent subsequence in A, you can find another one for each point by shifting the points in A. For instance, given $a_1, a_2 \in A$, one has $a_1 + a_2 \in A$.