Problem of the Week

09/21/2009 to 10/02/2009

The lattice points of the first quadrant are numbered as shown in the diagram, starting with the first lattice point lying at (0,0). For example, the 9^{th} lattice point is $(1, 2)$, while the 97^{th} lattice point is $(8, 5)$. Determine, with proof, the 2009^{th} lattice point in this scheme.

36	○	○	○	○	○	○	○
22	35	○	○	○	○	○	○
21	23	34	○	○	○	○	○
11	20	24	33	○	○	○	○
10	12	19	25	32	○	○	○
4	9	13	18	26	31	○	○
3	5	8	14	17	27	30	○
1	2	6	7	15	16	28	29

Solution: The correct point is $(55, 7)^*$.

Considering the diagonals (1), (2,3), (4,5,6), and so on, we see that the i^{th} diagonal contains i elements. Thus, we first look for i such that $1 + 2 + 3 + \cdots + i < 2009 < 1 + 2 + 3 + \cdots + i + i + 1$, that is,

$$\frac{i(i+1)}{2} < 2009 < \frac{(i+1)(i+2)}{2}.$$

When $i = 62$ this inequality is satisfied, and so 2009 lies in the 63^{rd} diagonal. Since 63 is odd, the top left element of this diagonal is $\frac{63(64)}{2} = 2016$, which lies on the point $(62, 0)$ in our lattice, and as $2016 - 2009 = 7$, the 2009^{th} lattice point lies 7 spaces to the right and below this point, thus it is at the point $(55, 7)$.

$*$ - When first posted, the problem statement had an error, and so some people solved the problem using that 1 was at the point $(1, 1)$ and thus 9 was at the point $(2, 3)$. In that way, the correct solution was $(56, 8)$.

Solutions for this problem were submitted by Matt Galla, Jeff Liese (Cal Poly, SLO), Matt Maly, Shah Purushottam, Jennifer Steele, Dennis Ugolini.