Problem of the Week

10/05/2009 to 10/16/2009

Given a real number x, let $\lfloor x \rfloor$ denote the largest integer n such that $n \leq x$. For example $\lfloor \pi \rfloor = 3$. Determine the number of real solutions to the equation

$$\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{3} \right\rfloor + \left\lfloor \frac{a}{5} \right\rfloor = a.$$

Solution: There are 30 real solutions to this equation.

Suppose x is a solution to the given equation. Then since $\lfloor \frac{x}{2} \rfloor$, $\lfloor \frac{x}{3} \rfloor$, and $\lfloor \frac{x}{5} \rfloor$ are integers, so is x. Thus, we can write $x = 30q + r$ with q, r integers and where $0 \leq r < 30$, that is, r is just the remainder of x when divided by 30. Then

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{5} \right\rfloor,$$

and substituting in for x we get

$$\begin{aligned} 30q + r &= \left\lfloor \frac{30q+r}{2} \right\rfloor + \left\lfloor \frac{30q+r}{3} \right\rfloor + \left\lfloor \frac{30q+r}{5} \right\rfloor \\ &= 15q + \left\lfloor \frac{r}{2} \right\rfloor + 10q + \left\lfloor \frac{r}{3} \right\rfloor + 6q + \left\lfloor \frac{r}{5} \right\rfloor \\ &= 31q + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{r}{3} \right\rfloor + \left\lfloor \frac{r}{5} \right\rfloor, \end{aligned}$$

and so $q = r - \lfloor \frac{r}{2} \rfloor + \lfloor \frac{r}{3} \rfloor + \lfloor \frac{r}{5} \rfloor$, that is, for every possible value of r we get a corresponding value of q which, in turn, gives us a solution x. Since there are 30 possible values of r, there are 30 solutions.

Solutions for this problem were submitted by Matt Galla, Mark Girard, Jeff Liese (Cal Poly, SLO), Matt Maly, Paul Nguyen, Jennifer Steele, David Stuck, and Dennis Ugolini.