Problem of the Week #11
1/27/2020 to 2/9/2020

Suppose we fill an \( n \times n \) matrix with the integers 1, 2, \ldots, \( n^2 \) in some order (each number used exactly once). Prove that there are two adjacent entries, in a row or a column, that differ by at least \( n \).

Solution: This turned out to be a tough one, and I give here a solution due to T.J. Gaffney:

Given a filled in \( n \times n \) grid, start shading the numbers in order. Say you’ve shaded cells with numbers 1 through \( k \). Stop when you first shade an entire column or row, and without loss of generality, let’s assume it’s a row. Then for each column, go up or down until you find the first non-shaded square, and mark that square with an “X,” which is possible because no column is yet completely shaded. You now have \( n \) cells marked with an “X.”

It must be that one of the marked cells has a value at least \( k + n \); if not, the pigeonhole principle would assign one of the \( n \) marked cells two of the values \( k + 1, k + 2, \ldots, k + n - 1 \), which is a contradiction. But the marked cell with the value at least \( k + n \) has a shaded neighbor (above or below it, by construction), which must have value at most \( k \), and so these two neighbors differ by at least \( n \).

In the case where the \( k \)-th square completes the first row and first column simultaneously, we can stop one short, and instead of shading the \( k \)-th square, mark it with an “X.” (Of course, its neighbors are shaded.) Then we proceed as described above to mark cells in the other columns, because the rest of that row must be shaded at that point.

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Otar Beridze (Georgia, the country), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, Maryland), Lincoln James (Chicago, IL), Steve King (Pullman, WA), Yann Michel (Paris, France), and Zurab Zakaradze (Georgia, the country).