Problem of the Week #2
9/2/2019 to 9/15/2019

A mathematician has 10 pairs of blue dress socks, ranging in 10 distinct shades from dark navy blue down to light blue. This mathematician has noticed that aside from wearing two socks from the same pair, he can also wear any two socks from adjacent shades without anyone but him being able to tell the difference—being a mathematician, he defines any two socks that are from the same or adjacent shades as an *acceptable pair*. One laundry day he is deep in thought working out a math problem while putting away his socks, and he randomly pairs them all together. What is the probability that all 10 pairs in this random pairing are acceptable pairings?

**Solution:** The probability is \( \frac{683}{19 \cdot 17 \cdot 15 \cdots 3} \approx 1.043 \times 10^{-6} \).

Let \( a(n) \) to be the number of pairings of \( n \) pairs of socks into acceptable pairings, e.g., \( a(1) = 1 \) and \( a(2) = 3 \). Further define the shades of the socks to be \( s_1 \) (lightest) through \( s_n \) (darkest). Then for \( n > 2 \), any sock with shade \( s_n \) must be paired with its mate or a sock of shade \( s_{n-1} \). If the two socks of shade \( s_n \) are paired together, then there are \( a(n-1) \) acceptable pairings of the remaining \( n-1 \) pairs. Similarly, if a sock of shade \( s_n \) is paired with a sock from the \( s_{n-1} \) pair, then their mates must also be paired together. There are two ways this can happen, and then \( a(n-2) \) acceptable pairings of the remaining \( n-2 \) pairs. Accordingly, \( a(n) = a(n-1) + 2a(n-2) \), which has characteristic polynomial \( \lambda^2 - \lambda - 2 = 0 \). The roots of this polynomial are \( \lambda = 2, -1 \), so that \( a(n) = c_1 2^n + c_2 (-1)^n \). Using our values of \( a(1) \) and \( a(2) \), we get that \( c_1 = 2/3 \) and \( c_2 = 1/3 \), giving

\[
a(n) = \frac{2^{n+1} + (-1)^n}{3}.
\]
Finally, if we wish to count all possible pairings, notice that any given sock can be paired with any of the other 19 socks; once those are paired any given remaining sock can be paired with any of the remaining 17 socks, and so on. Hence, the total number of pairings is $19 \cdot 17 \cdot 15 \cdots 3$, and thus the desired probability is

$$\frac{a(10)}{19 \cdot 17 \cdot 15 \cdots 3} = \frac{683}{19 \cdot 17 \cdot 15 \cdots 3}.$$  

Solutions for this problem were submitted by Anna Bown (Austin, TX), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, Maryland), Ben Jeffers (TU), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Lukas Klawuhn (Germany), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), Benjamin Phillipsbaum (Bothell, WA), A. Teitelman (Israel), Michael Tomaine (Bellevue, WA), Dennis Ugolini (TU), and Zurab Zakaradze (Georgia, the country).