Problem of the Week #3
9/16/2019 to 9/29/2019

Find all solutions \((x, y, z)\) to

\[
x^{y/z} = y^{z/x} = z^{x/y},
\]

where \(x\), \(y\), and \(z\) are positive real numbers.

Solution: It is clear that \((x, x, x)\), where \(x\) is any positive number, is a solution, and we will show that these are the only solutions. We first make the following three notes.

i. If \(a\) and \(b\) are positive numbers, then \(a^b < 1\) if and only if \(a < 1\). This gives that if any of the numbers \(x\), \(y\), or \(z\) is less than 1, then all three of these numbers are less than 1.

ii. Given a solution \((x, y, z)\), \((1/x, 1/z, 1/y)\) is also a solution. Indeed, we deduce that \((x^{y/z})^{-1} = (z^{x/y})^{-1} = (y^{z/x})^{-1}\). Hence, we may assume \(x, y, z \geq 1\).

iii. Because of symmetry, if \((x, y, z)\) is a solution, then any permutation of \((x, y, z)\) is also a solution. So, we may assume that \(x, y \leq z\).

Using iii.,

\[
x \geq x^{y/z} = y^{z/x} \geq y,
\]

so \(x/y \geq 1\). This gives that

\[
x \geq x^{y/z} = z^{x/y} \geq z,
\]
giving us that \(x = z\). Substituting these into our original equations, we now have \(x^{y/x} = y = x^{x/y}\), which occurs only if \(x = y\). Thus, \(x = y = z\).
Solutions for this problem were submitted by Colin Bown (Austin, TX), Phil Boyd (Manchester, England), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, Maryland), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), Benjamin Phillabaum (Bothell, WA), Luciano Santos (Portugal), and Zurab Zakaradze (Georgia, the country).