Problem of the Week #6
10/28/2019 to 11/10/2019

With justification, find the value of

$$\sum_{n=1}^{\infty} \left[ \sin^2 \left( \frac{\pi j}{2^n} \right) \right]^2,$$

where $j$ is a non-zero integer.

Solution: This sum has a value of 1 for any integer $j \neq 0$.

Computing the $n$-th partial sum gives

$$\sum_{n=1}^{N} \left[ \frac{\sin^2 \left( \frac{\pi j}{2^n} \right)}{\pi j/2^n} \right]^2 = \sum_{n=1}^{N} \frac{\sin^2 \left( \frac{\pi j}{2^n} \right) \left( 1 - \cos^2 \left( \frac{\pi j}{2^n} \right) \right)}{\frac{\pi j}{2^n} \frac{(\pi j)^2}{4^n}},$$

since $\sin^2(x) = 1 - \cos^2(x)$

$$= \sum_{n=1}^{N} \frac{\sin^2 \left( \frac{\pi j}{2^n} \right) - \sin^2 \left( \frac{\pi j}{2^n} \right) \cos^2 \left( \frac{\pi j}{2^n} \right)}{\frac{\pi j}{2^n} \frac{(\pi j)^2}{4^n}}$$

$$= \sum_{n=1}^{N} \frac{\sin^2 \left( \frac{\pi j}{2^n} \right) - \frac{1}{4} \sin^2 \left( \frac{\pi j}{2^{n-1}} \right)}{\frac{\pi j}{2^n} \frac{(\pi j)^2}{4^n}},$$

since $\sin(2x) = \frac{1}{2} \sin(x) \cos(x)$

$$= \sum_{n=1}^{N} \left[ \left( \frac{\sin \left( \frac{\pi j}{2^n} \right)}{\frac{\pi j}{2^n}} \right)^2 - \left( \frac{\sin \left( \frac{\pi j}{2^{n-1}} \right)}{\frac{\pi j}{2^{n-1}}} \right)^2 \right]$$

(which telescopes)

$$= \left( \frac{\sin \left( \frac{\pi j}{2^N} \right)}{\frac{\pi j}{2^N}} \right)^2 - \left( \frac{\sin \left( \frac{\pi j}{2^{N-1}} \right)}{\frac{\pi j}{2^{N-1}}} \right)^2$$

$$= \left( \frac{\sin \left( \frac{\pi j}{2^N} \right)}{\frac{\pi j}{2^N}} \right)^2,$$

since $\sin(\pi j) = 0$ for any integer $j$.

Thus,

$$\sum_{n=1}^{\infty} \left[ \frac{\sin^2 \left( \frac{\pi j}{2^n} \right)}{\pi j/2^n} \right]^2 = \lim_{N \to \infty} \left( \frac{\sin \left( \frac{\pi j}{2^N} \right)}{\frac{\pi j}{2^N}} \right)^2 = 1^2 = 1.$$
Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Otar Beridze (Georgia, the country), Phil Boyd (Manchester, England), Suliko Bolkvadze (Georgia, the country), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, Maryland), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), Benjamin Phillabaum (Bothell, WA), and Zurab Zakaradze (Georgia, the country).