Problem of the Week #14  
3/22/2021 to 4/4/2021

For a real number $z$, let $\lfloor z \rfloor$ denote the nearest integer to $z$ which is less than or equal to $z$, and define $\{z\} = z - \lfloor z \rfloor$. For any $k > 0$, find the value of

$$\int_0^1 \int_0^1 \left\{ \left( \frac{1}{x} \right)^k - \left( \frac{1}{y} \right)^k \right\} \, dx \, dy.$$ 

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**Solution:** The integral evaluates to $1/2$.

Let $I$ denote the desired value and notice that for any non-integer real number, $z$, $\{z\} + \{-z\} = 1$. By symmetry we have that

$$I = \int_0^1 \int_0^1 \left\{ \left( \frac{1}{x} \right)^k - \left( \frac{1}{y} \right)^k \right\} \, dx \, dy = \int_0^1 \int_0^1 \left\{ \left( \frac{1}{y} \right)^k - \left( \frac{1}{x} \right)^k \right\} \, dx \, dy.$$ 

Thus,

$$I = \frac{1}{2} (I + I)$$

$$= \frac{1}{2} \left[ \int_0^1 \int_0^1 \left\{ \left( \frac{1}{x} \right)^k - \left( \frac{1}{y} \right)^k \right\} \, dx \, dy + \int_0^1 \int_0^1 \left\{ \left( \frac{1}{y} \right)^k - \left( \frac{1}{x} \right)^k \right\} \, dx \, dy \right]$$

$$= \frac{1}{2} \int_0^1 \int_0^1 \left[ \left\{ \left( \frac{1}{x} \right)^k - \left( \frac{1}{y} \right)^k \right\} + \left\{ \left( \frac{1}{y} \right)^k - \left( \frac{1}{x} \right)^k \right\} \right] \, dx \, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^1 1 \, dx \, dy \quad \text{(see note on the next page)}$$

$$= \frac{1}{2}.$$
Note: There is definitely some “rigor” here that is missing, namely, why we don’t have to care about the fact that the integrand is (infinitely) sometimes 0. Some problem solvers did go through the details, but basically—and in an extremely non-technical wording—it’s just not zero all that often relative to how often it’s 1.

Solutions for this problem were submitted by Phil Boyd (Manchester, England), M.V. Channakeshava (India), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, MD), Lincoln James (Austin, TX), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).