Problem of the Week #4
10/5/2020 to 10/18/2020

The sequence of integers \( u_0, u_1, u_2, u_3, \ldots \) satisfies \( u_0 = 1 \) and

\[ u_{n+1}u_{n-1} = ku_n \quad \text{for each} \quad n \geq 1, \]

where \( k \) is some fixed positive integer. If \( u_{2000} = 2000 \), determine all possible values of \( k \).

Solution: The possible values of \( k \) are 2000, 1000, 500, 400, 200, and 100.

Rewrite the recursion so that \( u_{n+1} = \frac{ku_n}{u_{n-1}} \) and then begin by finding the first few terms of the sequence. To do this, we must first pick an integer value of \( u_1 \), say \( x \).

\[
\begin{align*}
  u_0 &= 1 \\
  u_1 &= x \\
  u_2 &= kx \\
  u_3 &= k^2 \\
  u_4 &= \frac{k^2}{x} \\
  u_5 &= \frac{k}{x} \\
  u_6 &= 1 = u_0 \\
  u_7 &= x = u_1
\end{align*}
\]

Since this sequence repeats every six terms and \( 2000 \equiv 2 \text{ mod } 6 \), we know that \( u_{2000} = u_2 \). Thus, we need \( u_2 = kx = 2000 \). Now, \( k \) and \( x \) are integers, and \( 2000 = 2^4 \cdot 5^3 \), so \( k \) and \( x \) are both of the form \( 2^a \cdot 5^b \). However, \( u_5 = \frac{k}{x} \) must also be an integer. The only such pairs \((k, x)\) satisfying all of these conditions are

\[(2000, 1), (1000, 2), (500, 4), (400, 5), (200, 10), \text{ and } (100, 20).\]

Thus, the possible values of \( k \) are 2000, 1000, 500, 400, 200, and 100.
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