A function $f$ is defined for all positive integers and satisfies

$$f(1) = 2020$$

and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n).$$

Calculate the exact value of $f(2020)$.

**Solution:** The solution is $\frac{2}{2021}$, which we will prove by induction on $n$. We first notice that

$$f(n) = \frac{f(1) + f(2) + \cdots + f(n-1)}{n^2 - 1}$$

and we use this to compute the first few values of $f$:

$$f(1) = 2020,$$

$$f(2) = \frac{1}{2^2 - 1} \cdot 2020,$$

$$f(3) = \frac{1}{2^2 - 1} \cdot \frac{2^2}{3^2 - 1} \cdot 2020,$$

$$f(4) = \frac{1}{2^2 - 1} \cdot \frac{2^2}{3^2 - 1} \cdot \frac{3^2}{4^2 - 1} \cdots 2020.$$

From this pattern it appears that

$$f(n) = 2020 \left( \prod_{j=2}^{n} \frac{(j-1)^2}{j^2 - 1} \right) = 2020 \left( \prod_{j=2}^{n} \frac{j-1}{j+1} \right) = \frac{2020 \cdot 2}{n(n+1)}.$$
For $n \geq 2$, let $P(n)$ be the statement that $f(n) = \frac{2 \cdot 2020}{n(n+1)}$. Then $P(2)$ is the statement that $f(2) = \frac{2 \cdot 2020}{2 \cdot 3} = \frac{f(1)}{2^2 - 1}$, which is true. Now assume $P(k)$ is true for some $k \geq 2$, that is

$$f(k) = \frac{2 \cdot 2020}{k(k+1)}.$$  

Then, by recursion,

$$f(k + 1) = \frac{f(1) + f(2) + \cdots + f(k)}{(k+1)^2 - 1}$$

$$= \frac{f(k)}{(k+1)^2 - 1} + \frac{f(1) + f(2) + \cdots + f(k-1)}{(k+1)^2 - 1}$$

$$= \frac{f(k)}{(k+1)^2 - 1} + \frac{f(k)(k^2 - 1)}{(k+1)^2 - 1} \cdot \frac{k^2 - 1}{k^2 - 1}$$

$$= \frac{f(k) \cdot k^2}{(k+1)^2 - 1}$$

$$= \frac{2020 \cdot 2 \cdot k}{k \cdot (k+1) \cdot (k+2)}$$ 

by the inductive hypothesis

$$= \frac{2020 \cdot 2}{(k + 1) \cdot (k + 2)},$$

which completes the induction. Thus

$$f(2020) = \frac{2020 \cdot 2}{2020(2020 + 1)} = \frac{2}{2021}.$$
Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Suliko Bolkvadze (Georgia, the country), Phil Boyd (Manchester, England), Matthew A. Brom (Troy, NY), M.V. Channakeshava (India), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, MD), Lincoln James (Austin, TX), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Jason Lee (Rockville, MD), Yann Michel (Paris, France), Surajit Rajagopal (India), Luciano Santos (Portugal), François Seguin (Amiens, France), A. Teitelman (Israel), and Zurab Zakaradze (Georgia, the country).