Problem of the Week #9
12/21/2020 to 1/24/2021

When $4444^{4444}$ is written in decimal notation, the sum of its digits is $A$. Let $B$ be the sum of the digits of $A$. Find the sum of the digits of $B$.

Solution: The sum of the digits of $B$ is 7.

Set $X = 4444^{4444}$. By the Mod 9 Divisibility Test, a number and the sum of its digits are congruent modulo 9:

$$4444 \equiv 4 + 4 + 4 + 4 \equiv 16 \equiv 7 \pmod{9},$$

giving

$$7^3 \equiv 1 \pmod{9} \Rightarrow 4444^3 \equiv 1 \pmod{9}.$$

As $4444 = 3 \cdot 1481 + 1$,

$$X \equiv 4444^{3(1481)} \cdot 4444 \equiv 1 \cdot 7 \equiv 7 \pmod{9}.$$

So we know that

$$X \equiv A \equiv B \equiv \text{the sum of the digits of } B \equiv 7 \pmod{9}.$$

Now,

$$\log_{10} X = 4444 \log_{10} 4444 < 4444 \log_{10} 10^4 = 4444 \cdot 4 = 17776.$$

This gives that $X$ has fewer than 17776 digits, and if they were all 9, then their sum would be $9 \cdot 17776 = 159984$. Thus, $A \leq 159984$. Of all numbers less than 159984, the one with largest sum is 99999, and so $B \leq 45$. Of all numbers less than or equal to 45, 39 has the largest sum of digits, and so the sum of the digits of $B$ is less than or equal to 12. The only number less than
or equal to 12 which is congruent to 7 modulo 9 is 7, and so the sum of the digits of $B$ is 7.

*Note:* By computer computation, one can show that $B = 16$, although I do not know of a good way to mathematically arrive at this result.

Solutions for this problem were submitted by Phil Boyd (Manchester, England), M.V. Channakeshava (India), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, MD), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).