Problem of the Week #13
2/28/2022 to 3/13/2022

A triangle has sides which form three consecutive integers, and the largest angle is twice the smallest angle. Find the cosine of the smallest angle.

Solution: The cosine of this angle is $\frac{3}{4}$.

If we let $\theta$ denote the smallest angle and $n-1$, $n$, $n+1$ denote the lengths of our three sides, then we know that the side opposite $\theta$ has length $n-1$ and the side opposite $2\theta$ has length $n+1$. Using the Law of Sines gives that

$$\frac{\sin(\theta)}{n-1} = \frac{\sin(2\theta)}{n+1} = \frac{2\sin(\theta)\cos(\theta)}{n+1} \Rightarrow \cos(\theta) = \frac{n+1}{2(n-1)}.$$ 

By the Law of Cosines we also have that

$$(n-1)^2 = (n+1)^2 + n^2 - 2n(n+1)\cos(\theta) \Rightarrow \cos(\theta) = \frac{n+4}{2(n+1)}.$$ 

Combining these two equations yields that $n = 5$, giving

$$\cos(\theta) = \frac{5+4}{2(5+1)} = \frac{3}{4}.$$ 

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Colin Bown (Austin, TX), Phil Boyd (Manchester, England), Matthew A. Brom (Troy, NY), M.V. Channakeshava (India), Ritwik Chaudhuri (India), Amelia Gibbs (Trinity undergraduate), Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Tin Lam (St. Louis, MO), Yann Michel (Paris, France), François Seguin (Amiens, France), A. Teitelman (Israel), and Zurab Zakaradze (Georgia, the country).