Problem of the Week #5
10/18/2021 to 10/31/2021

For any \( n = 1, 2, 3, \ldots \), find the value of

\[
\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \left\lfloor \frac{n+4}{8} \right\rfloor + \left\lfloor \frac{n+8}{16} \right\rfloor + \cdots,
\]

where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \).

Solution: The sum is always \( n \), and we offer two solutions of this fact.*

Solution 1. Write the binary representation of \( n \), that is,

\[
n = b_0 2^0 + b_1 2^1 + b_2 2^2 + b_3 2^3 + b_4 2^4 + \cdots,\]

where each \( b_i \in \{0, 1\} \). Then

\[
\left\lfloor \frac{n+1}{2} \right\rfloor = b_0 + b_1 + b_2 + b_3 2 + b_4 2^2 + \cdots \quad (1)
\]

\[
\left\lfloor \frac{n+2}{4} \right\rfloor = b_1 + b_2 + b_3 2 + b_4 2^2 + \cdots \quad (2)
\]

\[
\left\lfloor \frac{n+4}{8} \right\rfloor = b_2 + b_3 + b_4 2 + \cdots \quad (3)
\]

\[
\left\lfloor \frac{n+4}{8} \right\rfloor = b_3 + b_4 + \cdots \quad (4)
\]

By adding all of the columns we get back the binary representation of \( n \), thus obtaining the desired result.

Solution 2. Notice that \( \left\lfloor \frac{n+1}{2} \right\rfloor \) counts the number of even numbers between 1 and \( n + 1 \), inclusive, but this is the same as the number of odd numbers between 1 and \( n \), inclusive. More generally, \( \left\lfloor \frac{n+2^{i-1}}{2^i} \right\rfloor \) counts the number of multiples of \( 2^i \) between 1 and \( n + 2^{i-1} \), inclusive, which is the same as counting those numbers between 1 and \( n \), inclusive, which are multiples of \( 2^{i-1} \) but not multiples of \( 2^i \). Thus, the given sum counts each number from 1 to \( n \) exactly once.

*There was a typo for a few days in the original problem, where the “or equal to” was omitted from the definition of the floor function. I include the “or equal to” here, but the problem could be done in similar fashion without it, and you simply end up with \( n - 1 \) as the value of the sum.
Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Judhajit Barma (India), Phil Boyd (Manchester, England), M.V. Channakeshava (India), Ong See Hai (Singapore), Rob Hill (Gambrills, MD), Vaishnavi Josyla (Frisco, TX), Lukas Klawuhn (Germany), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), João Sant’Ana (Portugal), Luciano Santos (Portugal), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).