Problem of the Week #6
11/1/2021 to 11/14/2021

We may in various ways insert parentheses into the following expression in order to turn it into a valid expression: \( 1 \div 2 \div 3 \div 4 \). For instance \( 1 \div (2 \div (3 \div 4)) = \frac{3}{8} \), while \( 1 \div ((2 \div 3) \div 4) = 6 \). If we consider the expression \( 1 \div 2 \div 3 \div 4 \div 5 \div 6 \div 7 \div 8 \div 9 \div 10 \div 11 \), what is the smallest integer that can be obtained by the valid insertion of parentheses into this expression.

Solution: The smallest integer is 77.

To see this, first note that for any valid placement of parentheses, the new expression is equal to \( \frac{1 \cdot a \cdot b \cdot c \cdots}{2 \cdot x \cdot y \cdot z \cdots} \), where each of \( a, b, c, \ldots, x, y, z, \ldots \) is a number from 3, 4, \ldots, 11, and each of these numbers appears exactly once. Since 7 and 11 are primes that are coprime to all the other integers, they must be in the numerator for the expression to be an integer. In this particular case, we can arrange all of the other factors to cancel, leaving \( 11 \cdot 7 = 77 \) as the smallest integer that can be formed. Indeed, one such expression that yields 77 is given here, although there are many others as well:

\[
(1 \div 2 \div 3 \div 4 \div 5) \div (6 \div 7 \div 8 \div 9 \div 10 \div 11) = 77.
\]

Note: If we replace 11 by another positive integer \( n \), then for any valid placement of parentheses the new expression is equal to \( \frac{1 \cdot a \cdot b \cdot c \cdots}{2 \cdot x \cdot y \cdot z \cdots} \), where each of \( a, b, c, \ldots, x, y, z, \ldots \) is a number from 3, 4, \ldots, \( n \), and each of these numbers appears exactly once. We may rewrite the parenthesized expression as

\[
\frac{1 \cdot a \cdot b \cdot c \cdots}{2 \cdot x \cdot y \cdot z \cdots} = \frac{(1 \cdot a \cdot b \cdot c \cdots)(2 \cdot x \cdot y \cdot z \cdots)}{(2 \cdot x \cdot y \cdot z \cdots)^2} = \frac{n!}{(2 \cdot x \cdot y \cdot z \cdots)^2},
\]

and then factor \( n! \) appropriately to determine the smallest integer value. For
example, in the case where \( n = 11 \), we first factor 11! as such:

\[
11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 = 11 \cdot 7 \cdot (2 \cdot 5 \cdot 8 \cdot 9)^2.
\]

It follows that the smallest positive integer we may obtain is

\[
\frac{11!}{(2 \cdot 5 \cdot 8 \cdot 9)^2} = 77.
\]

Solutions for this problem were submitted by Phil Boyd (Manchester, England), M.V. Channakeshava (India), Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Vaishnavi Josyula (Frisco, TX), Lukas Klawuhn (Germany), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), Benjamin Phillabaum (Lafayette, IN), Luciano Santos (Portugal), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).