Problem of the Week #9  
12/13/2021 to 1/16/2022

Two wizards get on a bus, and one says to the other, “I have a positive number of children each of whom is a positive integer number of years old. The sum of their ages is the number of this bus and the product of their ages is my age.” To this the second wizard replies, “Perhaps if you told me your age and the number of children, I could work out the individual ages.” The first wizard replies, “No, you could not.” Then the second wizard says, “Now I know your age.” What is the number of the bus? Note that you should assume wizards always reason perfectly, can have any number of children, and can be any positive integer years old.

Note: This problem was due to the late, great John Conway, who passed away in 2020.

Solution: The bus number is 12.

To begin, define a number to be ambiguous if there are two different partitions of equal size whose summand product is the same. For example, 12 is ambiguous since

\[ 12 = 2 + 2 + 2 + 6 = 1 + 3 + 4 + 4 \quad \text{and} \quad 2 \cdot 2 \cdot 2 \cdot 6 = 1 \cdot 3 \cdot 4 \cdot 4 = 48. \]

Next, define a number to be doubly ambiguous if there are two different pairs partitions of whose summand product is the same. For example, 13 is doubly ambiguous since

\[ 13 = 1 + 6 + 6 = 1 + 4 + 9 \quad \text{with} \quad 1 \cdot 6 \cdot 6 = 1 \cdot 4 \cdot 9 = 36 \quad \text{and} \]

\[ 13 = 1 + 1 + 3 + 4 + 4 = 1 + 2 + 2 + 6 \quad \text{with} \quad 1 \cdot 2 \cdot 2 \cdot 6 = 1 \cdot 1 \cdot 3 \cdot 4 \cdot 4 = 48, \]

while one can verify that 12 is not doubly ambiguous. With these two definition in hand, we see that the first wizard’s response to the second wizard’s
first reply is equivalent to “The bus number is ambiguous”; however, the second wizard’s statement that he knows the first wizard’s age is equivalent to “The bus number is not doubly ambiguous.” We see by our example that if a number $n$ is ambiguous, then so is $n + 1$ (we simply add a summand of 1 to each partition of $n$, and this won’t change the corresponding products). Similarly, if $n$ is doubly ambiguous, so is $n + 1$. A check of the partitions of $n = 1, 2, \ldots, 11$, shows that 11 is not ambiguous, giving that the only ambiguous number which is not doubly ambiguous is $n = 12$.

Solutions for this problem were submitted by Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Tin Lam (St. Louis, MO), Yann Michel (Paris, France), Luciano Santos (Portugal), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).