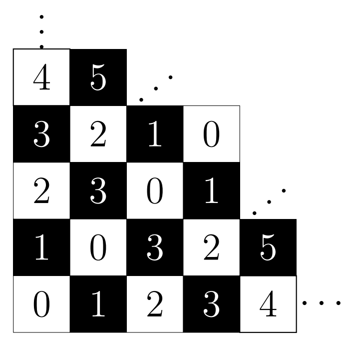
Summer Problem of 2022

5/16/2022 to 8/28/2022

The squares of an infinite chessboard are numbered successively as follows: in the bottom lefthand corner we place a 0, and then in every other square we place the smallest nonnegative integer that does not appear to its left in the same row or below it in the same column (see the figure below). What number appears at the intersection of the 2022nd row and the 500th column?
(Assume the columns are labeled from left-to-right and top-to-bottom starting with 1, i.e., 0 is at the intersection of the 1st row and 1st column.)



Solution: Let $\oplus$ denote the *Nim sum* operator defined as follows: if $x_k \cdots x_1 x_0$ and $y_k \cdots y_1 y_0$ are the binary representations of x and y, respectively, then $x \oplus y = z = z_k \cdots z_1 z_0$, where $z_i = 0$ if $x_i = y_i$ and $z_i = 1$ if $x_i \neq y_i$. For example,

$$11 \oplus 23 = (01011) \oplus (10111) = (10100) = 20.$$

We claim that the number in the intersection of the $(x+1)$st row and $(y+1)$st column is $x \oplus y$, and we need the following facts that all follow from the definition of $\oplus$: $x \oplus 0 = x$, $x \oplus x = 0$, $x \oplus y = y \oplus x$, and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

First note that since $0 \oplus 0 = 0$, our claim is correct for the intersection of the 1st row and 1st column.

Second, we have that $x \oplus y \neq u \oplus y$ for any $u < x$. To see this, suppose that some u exists such that $x \oplus y \neq u \oplus y$.

$$\begin{aligned}(x \oplus y) \oplus y = (u \oplus y) \oplus y &\Rightarrow x \oplus (y \oplus y) = u \oplus (y \oplus y) \\ &\Rightarrow x \oplus 0 = u \oplus 0 \\ &\Rightarrow x = u, \text{ a contradiction.}\end{aligned}$$

Similarly, $x \oplus y \neq x \oplus v$ for any $v < y$. However, it is indeed the case that if there is a number $w < x \oplus y$, then $w = u \oplus y$ for some $u < x$ or $w = x \oplus v$ for some $v < y$. We know that $x \oplus y = z_k \cdots z_1 z_0$, so assume $w_k \cdots w_1 w_0$ is the binary representation of w. Since $x \oplus y > w$, there exists a greatest m such that $z_m = 1$ and $w_m = 0$, giving that $x_m \neq y_m$. Suppose $x_m = 1, y_m = 0$. Then $y_m + z_m = 1 > y_m + w_m = 0$, but $y_j + z_j = y_j + w_j$ for all $m < j \leq k$, so $y \oplus w < y \oplus z$. However, $y \oplus z = y \oplus (y \oplus x) = x$, so $y \oplus w < x$. Indeed, we can now let $u = y \oplus w < x$, and then $u \oplus y = w$. By letting $x_m = 0$ and $y_m = 1$ in the previous argument, we would arrive at the fact that $w = x \oplus v$ for some $v < y$. Thus, our proof is complete, so we need only find the Nim sum of the binary representations of $2022 - 1 = 2021 = (11111100101)$ and $500 - 1 = 499 = (111110011)$:

$$2021 \oplus 499 = (11000010110) = 1558.$$

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), M.V. Channakeshava (India), Ritwik Chaudhuri (India), Evan Fu (Beaverton, OR), T.J. Gaffney (Las Vegas, NV), Rob Hill (Gambrills, MD), Vaishnavi Josyula (Frisco, TX), Lukas Klawuhn (Germany), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), Benjamin Phillabaum (Lafayette, IN), François Seguin (Amiens, France), Hicham Selmouni (Paris, France), and Zurab Zakaradze (Georgia, the country).