Jane plays a game with \( n > 1 \) marbles in the following manner: To start, Jane will divide the \( n \) marbles into two nonempty sets of size \( x \) and \( y \), that is, \( n = x + y \), and she will get \( xy \) points. If \( x > 1 \), then she further divides that subset into smaller sets of size \( x_1 \) and \( x_2 \) and \( x_1 x_2 \) points are awarded, and similarly so if \( y > 1 \). Jane continues in this way, adding her new points to her previous total until all subsets have size 1. What is the maximum number of points that Jane can get?

**Solution:** Jane’s maximum score is \( \frac{n(n-1)}{2} \), and indeed, Jane gets the same score regardless of her strategy.

Let \( P(N) \) be the statement that if Jane starts with \( N \) marbles she ends up with \( N(N-1)/2 \) points. Then \( P(2) \) is the statement that when Jane starts with 2 marbles she ends up with \( 2(2-1)/2 = 1 \) point, which is true. So assume that \( P(M) \) is true for \( M = 2, 3, \ldots, N-1 \), and we want to show that \( P(N) \) is true. Supposing Jane takes a pile of \( N \) marbles and splits them into two non-empty piles, one of size \( x \) and one of size \( y \), i.e., \( N = x + y \). Then we have that the point total Jane gets from this splitting the marbles in this way would be \( xy \) plus all subsequent points, which would be

\[
xy + \text{all points from the pile of } x \text{ marbles} + \text{all points from the pile of } y \text{ marbles}.
\]

By the induction hypothesis, since \( x, y \in \{2, 3, \ldots N - 1\} \), we have that this
point total is

\[
x y + \frac{x(x - 1)}{2} + \frac{y(y - 1)}{2} = xy + \frac{x^2 - x + y^2 - y}{2} = \frac{x^2 + 2xy + y^2 - (x + y)}{2} = \frac{(x + y)^2 - (x + y)}{2} = \frac{N^2 - N}{2} = \frac{(N - 1)N}{2}.
\]

Since \( P(N) \) is true whenever \( P(M) \) is true for \( M < N \), \( P(N) \) is true for all \( N > 2 \). Thus, Jane’s point total is always \( N(N - 1)/2 \) whenever she begins with \( N \) marbles.

*Note:* Many problem solvers pointed out that they were surprised by the fact that Jane’s score is fixed, regardless of her strategy, and I had the same reaction the first time I solved this problem!

Solutions for this problem were submitted by Colin Bown (Austin, TX), M.V. Channakeshava (India), Ritwik Chaudhuri (India), Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Lukas Klawuhn (Germany), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).