Problem of the Week #17

4/24/2023 to 5/7/2023

Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n),$$

where $\ln$ denotes the natural logarithm function. Find the value of

$$x_0 + x_1 + x_2 + \cdots .$$

Solution: The value of the is sum is $e - 1$.

First let $f(x) = e^x - x$, which has derivative $f'(x) = e^x - 1 > 0$ for $x > 0$, so f is strictly increasing on $(0, \infty)$. Since $x_0 = 1$, and $\ln(x)$ is an increasing function, we have that $\ln(f(x_n)) = x_n > 0$ for all $n \geq 0$. Now,

$$x_{n+1} = \ln(e^{x_n} - x_n) \Rightarrow e^{x_{n+1}} = e^{\ln(e^{x_n} - x_n)} \Rightarrow x_n = e^{x_n} - e^{x_{n+1}}.$$

Since $x_n > 0$, we have $e^{x_n} > e^{x_{n+1}}$, so $x_n > x_{n+1}$, giving that the sequence $x_0, x_1, x_2, \ldots$ is decreasing, and as it is also bounded below by 0, it converges to some limit L. We may then compute L as

$$L = \lim_{n \to \infty} x_n = \lim_{n \to \infty} (e^{x_n} - e^{x_{n+1}}) = e^L - e^L = 0,$$

which gives that

$$\lim_{n \to \infty} e^{x_n} = e^L = e^0 = 1.$$

Computing the partial sum of the x_n's yields

$$\begin{aligned} x_0 + x_1 + \cdots + x_n &= (e^{x_0} - e^{x_1}) + (e^{x_1} - e^{x_2}) + \cdots + (e^{x_n} - e^{x_{n+1}}) \\ &= e^{x_0} - e^{x_{n+1}}, \text{ since the sum telescopes} \\ &= e - e^{x_{n+1}} \end{aligned}$$

Finally, taking limits of both sides we obtain $x_0 + x_1 + x_2 + \cdots = e - 1$.

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Colin Bown (Austin, TX), M.V. Channakeshava (India), Quentin Finn (alum), Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Yann Michel (Paris, France), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).