Problem of the Week #2

9/12/2022 to 9/25/2022

Pick $n \geq 3$, and divide a circle into $2n$ arcs by placing $2n$ points on its boundary subject to the following conditions:

i. **there are only three allowable arc lengths;**

ii. **no adjacent arcs have the same length;**

iii. **vertices are colored alternately red ad blue.**

Prove that the n-gon formed by the red vertices and the n-gon formed by the blue vertices have the same perimeter *and* the same area as each other.

Solution: Call the n-gon with red vertices R and the the one with blue vertices B. Let the arc lengths be denoted by a, b, and c, and suppose there are x, y, and z of each length, respectively. Then $x + y + z = 2n$. Focusing first on R, each side is subtended by an arc of length $a + b$, $a + c$, or $b + c$, and of these, x contain an arc of length a, so there are $n - x$ arcs of length $b + c$. Similarly, the number of arcs of length $a + b$ is $n - z$ and the number of arcs of length $a + c$ is $n - y$. Notice the exact same is true of B. Since a given arc length determines the corresponding side length, R and B must have the same perimeter.

For the area, a given arc length also fixes the amount of area between the side of the polygon and the circle, and thus the amount of area lying outside R but inside the circle is the same as that for B, giving that R and B have the same area.

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Yann Michel

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