Problem of the Week #2
9/12/2022 to 9/25/2022

Pick \( n \geq 3 \), and divide a circle into \( 2n \) arcs by placing \( 2n \) points on its boundary subject to the following conditions:

i. there are only three allowable arc lengths;

ii. no adjacent arcs have the same length;

iii. vertices are colored alternately red and blue.

Prove that the \( n \)-gon formed by the red vertices and the \( n \)-gon formed by the blue vertices have the same perimeter and the same area as each other.

Solution: Call the \( n \)-gon with red vertices \( R \) and the one with blue vertices \( B \). Let the arc lengths be denoted by \( a, b, \) and \( c \), and suppose there are \( x, y, \) and \( z \) of each length, respectively. Then \( x + y + z = 2n \). Focusing first on \( R \), each side is subtended by an arc of length \( a + b, a + c, \) or \( b + c \), and of these, \( x \) contain an arc of length \( a \), so there are \( n - x \) arcs of length \( b + c \). Similarly, the number of arcs of length \( a + b \) is \( n - z \) and the number of arcs of length \( a + c \) is \( n - y \). Notice the exact same is true of \( B \). Since a given arc length determines the corresponding side length, \( R \) and \( B \) must have the same perimeter.

For the area, a given arc length also fixes the amount of area between the side of the polygon and the circle, and thus the amount of area lying outside \( R \) but inside the circle is the same as that for \( B \), giving that \( R \) and \( B \) have the same area.

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