Problem of the Week #5

10/24/2022 to 11/6/2022

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let $y_1, y_2, \ldots, y_n$ be any rearrangement of those numbers. Prove that

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n} \geq n.$$

Solution: This is an application of the arithmetic-geometric mean inequality (AM-GM), which states that for any positive, real numbers $a_1, a_2, \ldots a_n$,

$$\frac{1}{n}(a_1 + a_2 + \cdots + a_n) \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

In our case, consider the n positive real numbers $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}$. Notice that since the sequences of x_i's and y_i's are rearrangements of one another,

$$\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \cdot \cdots \cdot \frac{x_n}{y_n} = 1.$$

Thus, by AM-GM,

$$\frac{1}{n}\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}\right) \geq \sqrt[n]{\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \cdot \cdots \cdot \frac{x_n}{y_n}} = \sqrt[n]{1} = 1.$$

Multiplying both sides by n yields the desired result.

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), M.V. Channakeshava (India), Ritwik Chaudhuri (India), Ruben Victor Cohen (Argentina), Evan Fu (Beaverton, OR), Ong See Hai (Singapore), Rob Hill (Gambrills, MD), Steve King (Pullman, WA), Hari Kishan (India), Lukas Klawuhn (Germany), Tengiz Kutchava (Georgia, the country), Tin Lam (St. Louis, MO), Yann Michel (Paris, France), François Seguin (Amiens, France), Hicham Selmouni (Paris, France), and Zurab Zakaradze (Georgia, the country).