Problem of the Week #6

11/7/2022 to 11/20/2022

Let $f(x)$ be a polynomial with integral coefficients, and further suppose there are two distinct points on the graph of f, say P and Q, with integral coordinates. Show that if the length of $\overline{PQ}$ is also integral, then $\overline{PQ}$ is parallel to the x-axis.

Solution: Write $f = a_k x^k + a_{k-1}x^{k-1} + \cdots a_1 x + a_0$, and suppose $P = (u, v)$ and $Q = (w, z)$ for some integers u, v, w, z. Note that

$$v - z = f(u) - f(w) = a_k(u^k - w^k) + a_{k-1}(u^{k-1} - w^{k-1}) + \cdots + (u - w),$$

and since $u - w$ divides $u^t - w^t$ for any positive integer t, we have that $u - w$ divides $v - z$, that is, $v - z = n(u - w)$ for some integer n. Now, if the length of $\overline{PQ} = m > 0$, then

$$\begin{aligned} m^2 &= (u-w)^2 + (v-z)^2 \\ &= (u-w)^2 + n^2(u-w)^2 \\ &= (n^2+1)(u-w)^2, \end{aligned}$$

which gives that $n^2 + 1$ must also be a perfect square, and also positive, since m is positive. However, this only occurs when $n = 0$, giving that $v = z$.

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), M.V. Channakeshava (India), Ritwik Chaudhuri (India), Evan Fu (Beaverton, OR), Rob Hill (Gambrills, MD), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Tin Lam (St. Louis, MO), Yann Michel (Paris, France), François Seguin (Amiens, France), Hicham Selmouni (Paris, France), and Zurab Zakaradze (Georgia, the country).