Let \( f(x) \) be a polynomial with integral coefficients, and further suppose there are two distinct points on the graph of \( f \), say \( P \) and \( Q \), with integral coordinates. Show that if the length of \( PQ \) is also integral, then \( PQ \) is parallel to the \( x \)-axis.

**Solution:** Write \( f = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 \), and suppose \( P = (u, v) \) and \( Q = (w, z) \) for some integers \( u, v, w, z \). Note that

\[
v - z = f(u) - f(w) = a_k (u^k - w^k) + a_{k-1} (u^{k-1} - w^{k-1}) + \cdots + (u - w),
\]

and since \( u - w \) divides \( u^t - w^t \) for any positive integer \( t \), we have that \( u - w \) divides \( v - z \), that is, \( v - z = n(u - w) \) for some integer \( n \). Now, if the length of \( PQ = m > 0 \), then

\[
m^2 = (u - w)^2 + (v - z)^2 = (u - w)^2 + n^2 (u - w)^2 = (n^2 + 1)(u - w)^2,
\]

which gives that \( n^2 + 1 \) must also be a perfect square, and also positive, since \( m \) is positive. However, this only occurs when \( n = 0 \), giving that \( v = z \).

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