Problem of the Week #7

11/21/2022 to 12/4/2022

Let $x = 3168$ **and** $y = 7a2b4c54995d458e1f9g6h3i17862$**, where** $abcdefghi$ **is some permutation of the digits** $1, 2, 3, 4, 5, 6, 7, 8, 9$**. How many such permtations exist such that the remainder of** y **upon division by** x **is exactly 2022?**

Solution: There are $9! = 362,880$ such permutations, that is, any permutation will work!

We are being asked to show that $y - 2022$ is divisisible by $3168 = 2^5 \cdot 9 \cdot 11$, that is 3168 divides $z = 7a2b4c54995d458e1f9g6h3i15840$.

First, note that 2^k divides an integer m whenever 2^k divies the last k digits of m. Since 2^5 divides 15840, this gives that 2^5 divides z.

Also, the sum of z's digits is

$$99 + a + b + c + d + e + f + g + h + i = 99 + 45 = 144,$$

which is divisible by 9, and so z is divisible by 9.

Finally, the alternating sum of z's digits is

$$72 - (a + b + c + 4 + 9 + d + 5 + e + f + g + g + i + 5 + 4) = 72 - (45 + 27) = 0,$$

which is divisible by 11.

Since 2^5, 9, and 11 are all relatively prime, z is dvisible by their product for any choice of $abcdefghi$.

Solutions for this problem were submitted by Ziad Aramouni (Lebanon), Tanay Arora (Beaverton, OR), Matthew A. Brom (Albany, NY), M.V. Channakeshava (India), Ritwik Chaudhuri (India), Ruben Victor Cohen (Argentina), Quentin Finn (alum), Rob Hill (Gambrills, MD), Hari Kishan (India), Tengiz Kutchava (Georgia, the country), Tin Lam (St. Louis, MO), Yann Michel (Paris, France), François Seguin (Amiens, France), Hicham Selmouni (Paris, France), Michael Tomaine (Bellevue, WA), and Zurab Zakaradze (Georgia, the country).