Problem of the Week #8

12/5/2022 to 12/18/2022

For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2022} c(n)c(n+2).$$

Solution: The sum has a value of -2.

Note that

$$\begin{aligned} c(2k+1)c(2k+3) &= (-1)^k c(k)(-1)^{k+1} c(k+1) \\ &= -c(k)c(k+1) \\ &= -c(2k)c(2k+2). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n=1}^{2022} c(n)c(n+2) &= c(1)c(3) + c(2022)c(2024) + \sum_{k=1}^{1010} (c(2k)c(2k+2) + c(2k+1)c(2k+3)) \\ &= c(1)c(3) + c(2022)c(2024). \end{aligned}$$

Applying the given rules, $c(3) = (-1)^1 c(1) = -1$, so that $c(1)c(3) = -1$. We can similarly reduce $c(2022)c(2024)$ to obtain its value of -1. Thus, the sum's value is

$$c(1)c(3) + c(2022)c(2024) = (-1) + (-1) = -2.$$

Note: This is (a slightly modified version of) Problem B1 from the 2013 Putnam Exam. The 2022 Putnam exam took place on Saturday, December 3rd, and I like to honor each year's exam with a problem from a previous exam.

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