Problem of the Week #8
12/5/2022 to 12/18/2022

For positive integers \( n \), let the numbers \( c(n) \) be determined by the rules \( c(1) = 1 \), \( c(2n) = c(n) \), and \( c(2n + 1) = (-1)^n c(n) \). Find the value of

\[
\sum_{n=1}^{2022} c(n)c(n + 2).
\]

**Solution:** The sum has a value of -2.

Note that

\[
c(2k + 1)c(2k + 3) = (-1)^k c(k)(-1)^{k+1} c(k + 1) = -c(k)c(k + 1) = -c(2k)c(2k + 2).
\]

It follows that

\[
\sum_{n=1}^{2022} c(n)c(n + 2) = c(1)c(3) + c(2022)c(2024) + \sum_{k=1}^{1010} (c(2k)c(2k + 2) + c(2k + 1)c(2k + 3))
\]

\[
= c(1)c(3) + c(2022)c(2024).
\]

Applying the given rules, \( c(3) = (-1)^1 c(1) = -1 \), so that \( c(1)c(3) = -1 \). We can similarly reduce \( c(2022)c(2024) \) to obtain its value of -1. Thus, the sum’s value is

\[
c(1)c(3) + c(2022)c(2024) = (-1) + (-1) = -2.
\]

**Note:** This is (a slightly modified version of) Problem B1 from the 2013 Putnam Exam. The 2022 Putnam exam took place on Saturday, December 3rd, and I like to honor each year’s exam with a problem from a previous exam.
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