Problem of the Week #9

12/19/2022 to 1/15/2023

On a circle are 999 points numbered 1 to 999. We wish to color each of the points with the colors blue, red, and green such that between any two points colored with the same color there are an even number of colored points differing from these two. In how many ways can we color these 999 points?

Note: Only a small number of people attmpeted this problem, and very few had the right answer. This one was definitely a harder problem than usual, and in addition, some problem solvers had issues determining which sorts of colorings were allowable. In particular, note the following two facts:

i. it is okay to have no differing colored points between two identically colored points, i.e., consecutive, idenitcal point coloring is allowed;

ii. we allow for a point and itself to have this property.

To illustrate these, here are the allowable colorings for circles with $n = 1$, 3, and 5 points, respectively:

i. $\underline{n = 1}$: $3 = 2^1 + 1$ colorings
B, R, G

ii. $\underline{n = 3}$: $9 = 2^3 + 1$ colorings
BBB, RRR, GGG, BRG, BGR, RBG, RGB, GBR, GRB

iii. $\underline{n = 5}$: $33 = 2^5 + 1$ colorings
BBBBB, RRRRR, GGGGG,
BBBRG, GBBBR, RGBBB, BRGBB, BBRGB,
plus the other 25 colorings where we permute the B's, R's, and G's in the row above

Solution: As the examples above suggest, there are $2^{999} + 1$ allowable colorings, and to prove this, let us first set aside the three monochromatic colorings. For the remaining colorings, we make two observations:

i. each coloring must contain each color at least once, and

ii. each color occurs an odd number of times.

From here, given any number x chosen from 1 to 999, define $f(x) = (b, r, g)$, where b denotes the numbers points, starting at point 1 and moving clockwise about the circle towards the point numbered x, until a B is encountered, and similarly for r and g. Note that if $x = 1$, we take that to mean that we are trying to go all the way around the circle. By construction one of b, r, or g is 0. Further, we can show that *exactly* one of the other coordinates of the triple is odd and whichever coordinate is odd for x will not be odd for $x + 1$. Now, given $f(x) = (b, r, g)$, define $g(x)$ to be α if b is odd, β if r is odd, and γ if g is odd. Then we have a bijection between the allowable colorings and n-length sequences of α's, β's, and γ's so that no consecutive letters are equal and the first and last letter differ.

Define t_n to be the number of n-length sequences of α's, β's, and γ's so that no consecutive letters are equal and the first and last letter differ, and define s_n to be the number of n-length sequences of α's, β's, and γ's so that no consecutive letters are equal and the first and last letter are equal. Then we see that any sequence counted by s_{n+1} is obtained by taking a sequence counted by t_n and adding the first letter to the end, i.e.,

$$s_{n+1} = t_n. \tag{1}$$

Similarly, any sequence counted by t_{n+1} may be obtained in two ways. First, we can take a sequence counted by t_n and add to the end the number that is in neither the first nor the n-th position. Second, we could take any sequence counted by s_n and add to it either of the two letters that are not in the n-th (and so also the first) position. Accordingly,

$$t_{n+1} = t_n + 2s_n. \tag{2}$$

By (1), $s_n = t_{n-1}$, and substituting this into (2) gives

$$t_{n+1} = t_n + 2t_{n-1}. \tag{3}$$

Solving this linear recurrence gives that $t_n = c_1 2^n + c_2(-1)^n$. Using our examples from the first page, and disregarding the monochromatic colorings, we know that $t_3 = 6$ and $t_5 = 30$, and so we obtain $c_1 = 1, c_2 = 2$. Thus, $t_{2n+1} = 2^{2n+1} - 2$, so that the number of allowable colorings is

$$3 + t_{999} = 3 + (2^{999} - 2) = 2^{999} + 1.$$

Solutions for this problem were submitted by Hari Kishan (India), T.J. Gaffney (Las Vegas, NV), Amelia Gibbs (TU), Rob Hill (Gambrills, MD), Tengiz Kutchava (Georgia, the country), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).