Summer Problem of 2023

5/8/2023 to 8/27/2023

Define $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, and for any subset K of $\mathbb{N}_0$, define the function $f_K(n)$ to be the number of ways to write $n = k_1 + k_2$ where $k_1, k_2 \in K$ and $k_1 < k_2$. For example, if $K = \{1, 2, 3, 5, 8, 13\}$, then $f_K(1) = 0$, while $f_K(8) = 1$, as $8 = 3 + 5$. Find, with justification, a way to partition $\mathbb{N}_0$ into two sets, A and B, such that $f_A(n) = f_B(n)$ for every *positive* integer n, or prove that no such partition exists.

Solution: It is possible to find such an A and B!

To convince ourselves that a partition exists, let's place 0 into A (since 0 must go somewhere). Then, 1 must go into B, or else, $f_A(1) = 1 \neq 0 = f_B(1)$. Going integer-by-integer, we would soon see that it *appears* we have

$$A = \{0, 3, 5, 6, 9, 10 \ldots\} \text{ and } B = \{1, 2, 4, 7, 8, \ldots\},$$

but it is neither clear (nor justified) that this is going to work forever, nor do we know that such a partition, if it does exist, is unique. Furthermore, it's not even clear what rule, if any, would place some integer m into A or B.

So, let's assume that a partition does exist, and define the following two generating functions:

$$\mathcal{A}(x) = \sum_{a \in A} x^a \text{ and } \mathcal{B}(x) = \sum_{b \in B} x^b.$$

Since A, B partition $\mathbb{N}_0$,

$$\mathcal{A}(x) + \mathcal{B}(x) = \sum_{n \in \mathbb{N}_0} x^n = \frac{1}{1-x}. \tag{1}$$

Next, notice that

$$\mathcal{A}^2(x) = \left(\sum_{a \in A} x^a\right)^2 = \sum_{\substack{a_1, a_2 \in A \\ a_1 \leq a_2}} x^{a_1 + a_2},$$

while

$$\mathcal{A}(x^2) = \sum_{a\in A} (x^2)^a = \sum_{a\in A} x^{2a}.$$

Combining these last two equations gives

$$\mathcal{A}^2(x) - \mathcal{A}(x^2) = \sum_{\substack{a_1,a_2\in A\\ a_1<a_2}} x^{a_1+a_2},$$

so that the only powers of x we will see in $\mathcal{A}^2(x) - \mathcal{A}(x^2)$ are those that can be obtained by adding two *distinct* elements from A. Moreover, every pair of distinct elements from A that add up to some $n \in \mathbb{N}_0$ will contribute to the coefficient of x^n, that is,

$$\mathcal{A}^2(x) - \mathcal{A}(x^2) = \sum_{n\in\mathbb{N}_0} f_A(n)x^n.$$

Applying the same logic to $\mathcal{B}(x)$, the problem statement gives us that

$$\begin{aligned}\mathcal{A}^2(x) - \mathcal{A}(x^2) = \mathcal{B}^2(x) - \mathcal{B}(x^2) \;&\Rightarrow\; \mathcal{A}^2(x) - \mathcal{B}^2(x) = \mathcal{A}(x^2) - \mathcal{B}(x^2)\\ &\Rightarrow\; (\mathcal{A}(x) - \mathcal{B}(x))(\mathcal{A}(x) + \mathcal{B}(x)) = \mathcal{A}(x^2) - \mathcal{B}(x^2)\\ &\Rightarrow\; \frac{(\mathcal{A}(x) - \mathcal{B}(x))}{1-x} = \mathcal{A}(x^2) - \mathcal{B}(x^2), \text{ by (1)}\\ &\Rightarrow\; \mathcal{A}(x) - \mathcal{B}(x) = (1-x)(\mathcal{A}(x^2) - \mathcal{B}(x^2)).\end{aligned}$$

Repeatedly applying this last equality to the right hand side gives us the following:

$$\mathcal{A}(x) - \mathcal{B}(x) = \left(\mathcal{A}(x^{2^k}) - \mathcal{B}(x^{2^k})\right) \cdot \prod_{i=0}^{k-1} \left(1 - x^{2^i}\right).$$

Since $0 \in A$, the first term of $\mathcal{A}(x^{2^i})$ is 1, whereas there is no constant term in $\mathcal{B}(x^{2^i})$, giving that

$$\mathcal{A}(x) - \mathcal{B}(x) = \lim_{k\to\infty} \left[\left(\mathcal{A}(x^{2^k}) - \mathcal{B}(x^{2^k})\right) \cdot \prod_{i=0}^{k-1} \left(1 - x^{2^i}\right)\right] = \prod_{i=0}^{\infty} \left(1 - x^{2^i}\right).$$

This product, when multiplied out, gives a positive coefficient in front of x^j whenever j is the sum of an even number of distinct powers of 2; similarly, we'll get a negative coefficient in front of x^j whenever j is the sum of an odd number of distinct powers of 2. Thus, A is the set of all elements of $\mathbb{N}_0$ with an even number of 1's in their binary representation, and B is the set of all elements of $\mathbb{N}_0$ with an odd number of 1's in their binary representation.

Solutions for the summer problem were submitted by Rob Hill (Gambrills, MD), Tengiz Kutchava (Georgia, the country), François Seguin (Amiens, France), and Zurab Zakaradze (Georgia, the country).