Some Combinatorial Properties of Poly-Stirling Numbers

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CombinaTexas 2008

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A Quick Outline

- What is a *poly-Stirling number*?
- A special case with some combinatorial interpretations.
- Some generalized results.

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Definitions

Suppose we are given a polynomial $p(x) = \sum_{i=1}^{s} a_i x^i$ where the a_i's are nonnegative integers.

Definition: We define $S(n, k, p)$ by the recursion $S(n+1, k, p) = S(n, k-1, p) + p(k)S(n, k, p)$ with initial conditions $S(0, 0, p) = 1$ and $S(n, k, p) = 0$ if $n < k \leq 0$. We call the number $S(n, k, p)$ a *poly-Stirling number of the second kind.*

Poly-Stirling Numbers

More Definitions

Definition: We define $s(n,k,p)$ by the recursion $s(n+1,k,p) = s(n,k-1,p) - p(n)s(n,k,p)$ with initial conditions $s(0,0,p) = 1$ and $s(n,k,p) = 0$ if $n < k \leq 0$. We call the number $s(n,k,p)$ a *poly-Stirling number of the first kind*.

Definition: We define $c(n,k,p) = (-1)^{n-k}s(n,k,p)$, which will satisfy a similar recursion and initial conditions. We call the number $c(n,k,p)$ a *signless poly-Stirling number of the first kind*.

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A Quick Recap

- $S(n+1,k,p) = S(n,k-1,p) + p(k)S(n,k,p)$
- $s(n+1,k,p) = s(n,k-1,p) - p(n)s(n,k,p)$
- $c(n+1,k,p) = c(n,k-1,p) + p(n)c(n,k,p)$

Note: These are exactly the standard Stirling numbers when $p(x) = x$. In the case where $p(x) = x^2$ we get the *central factorial numbers*[1].

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[1] - See Stanley's *Enumerative Combinatorics, Vol. II* or Riordan's *Combinatorial Identities.*

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x^m-Stirling Numbers

We first consider the case where $p(x) = x^m$ for some $m \in \mathbb{N}$.

- $S(n+1,k,p) = S(n,k-1,p) + k^m S(n,k,p)$
- $s(n+1,k,p) = s(n,k-1,p) - n^m s(n,k,p)$
- $c(n+1,k,p) = c(n,k-1,p) + n^m c(n,k,p)$

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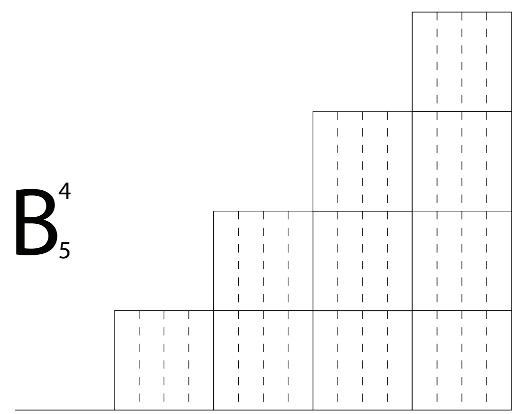
Poly-Stirling Numbers

Rook Board Theoretic Interpretations

Given $m, n \in \mathbb{N}$, define $\mathcal{B}_n^m$ to be a rook board with column heights, from left to right, of $0, 1, 2, \ldots, n-1$ with each column partitioned into m subcolumns.



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Attacking Rook Theoretic Interpretations

Attacking rook placement rules in the board $\mathcal{B}_n^m$ are as follows:

- If a rook is placed in one subcolumn of a column, C, then a rook must be placed in every subcolumn of C, i.e., every column contains either 0 rooks or m rooks.
- Each subcolumn may only contain one rook.
- A rook cancels all cells to its right in its respective subcolumn and all previously uncanceled cells directly below itself in its subcolumn.

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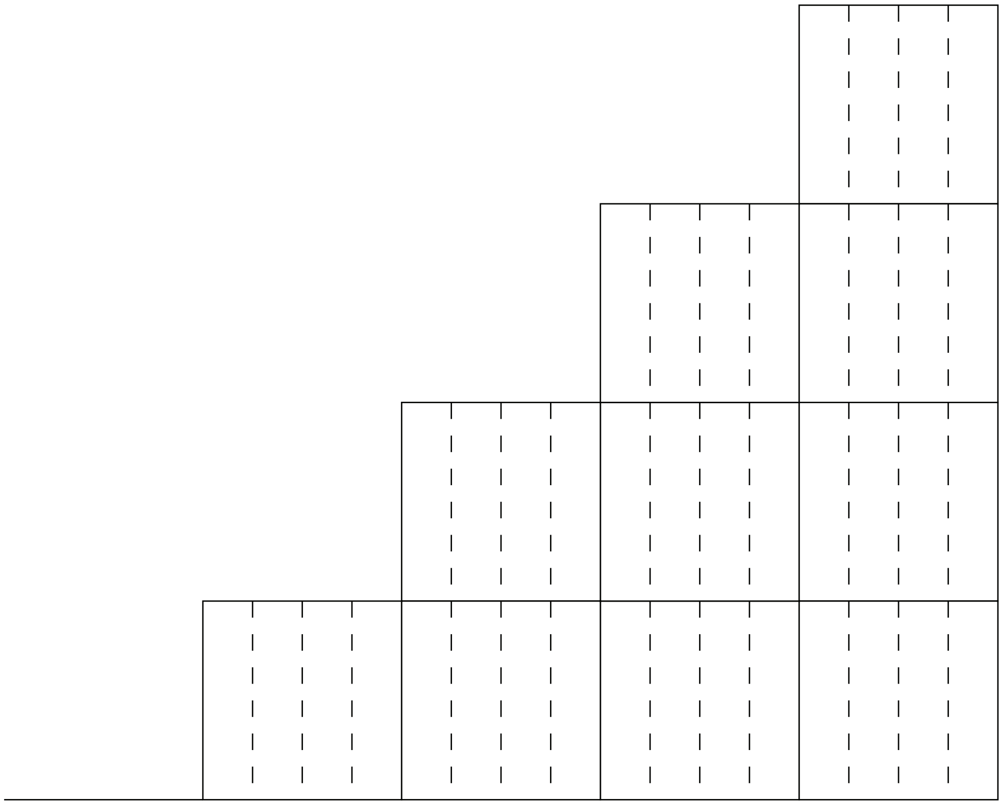
Poly-Stirling Numbers

Example of an Attacking Rook Placement in B_5^4



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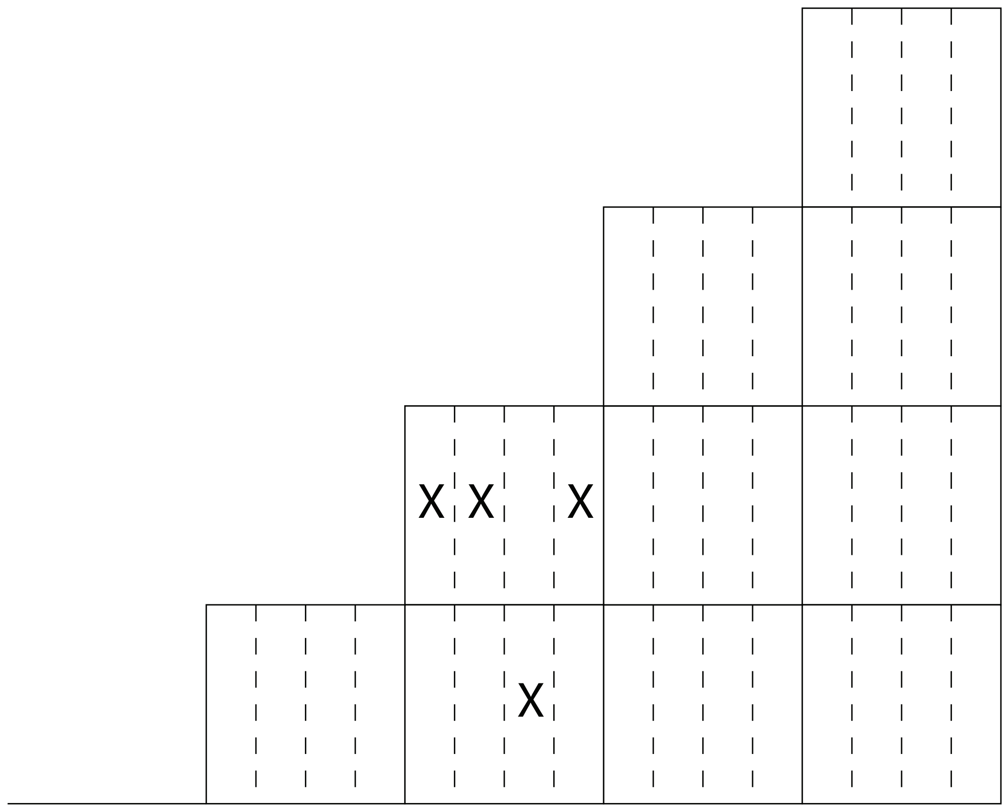

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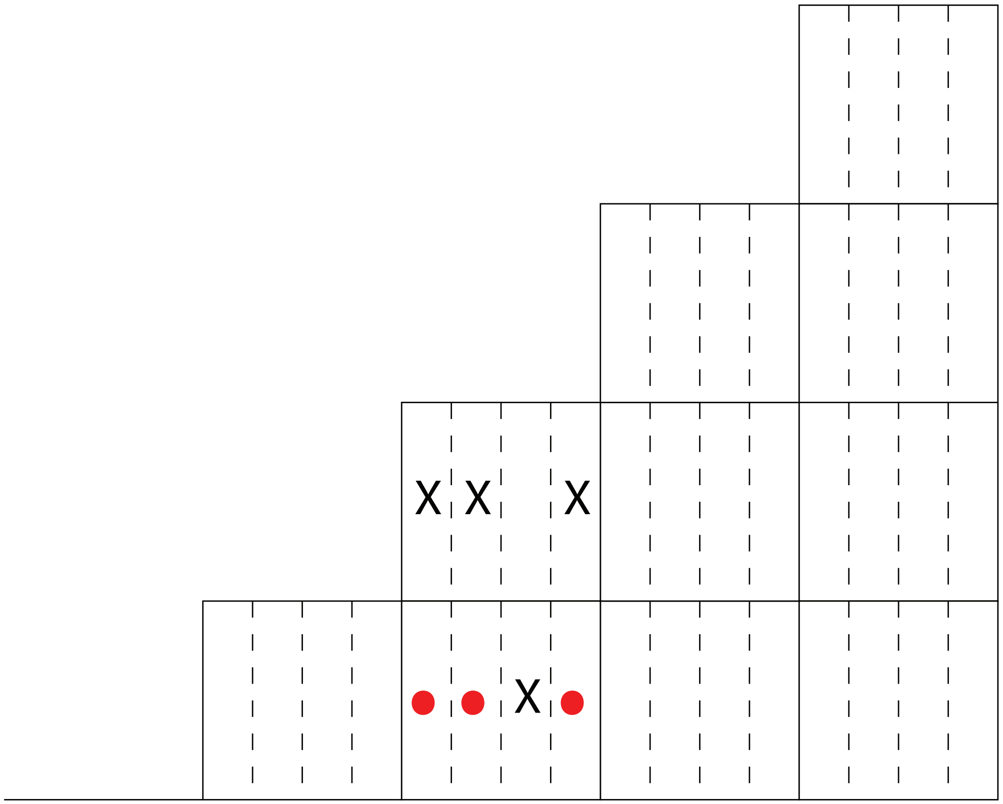

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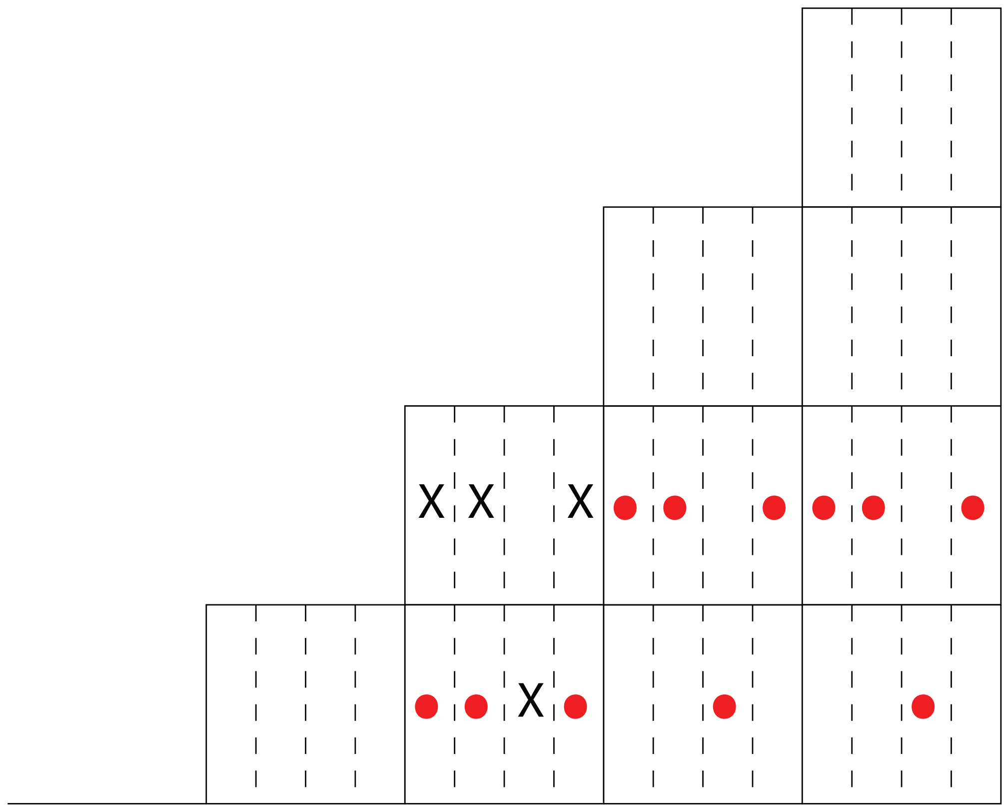

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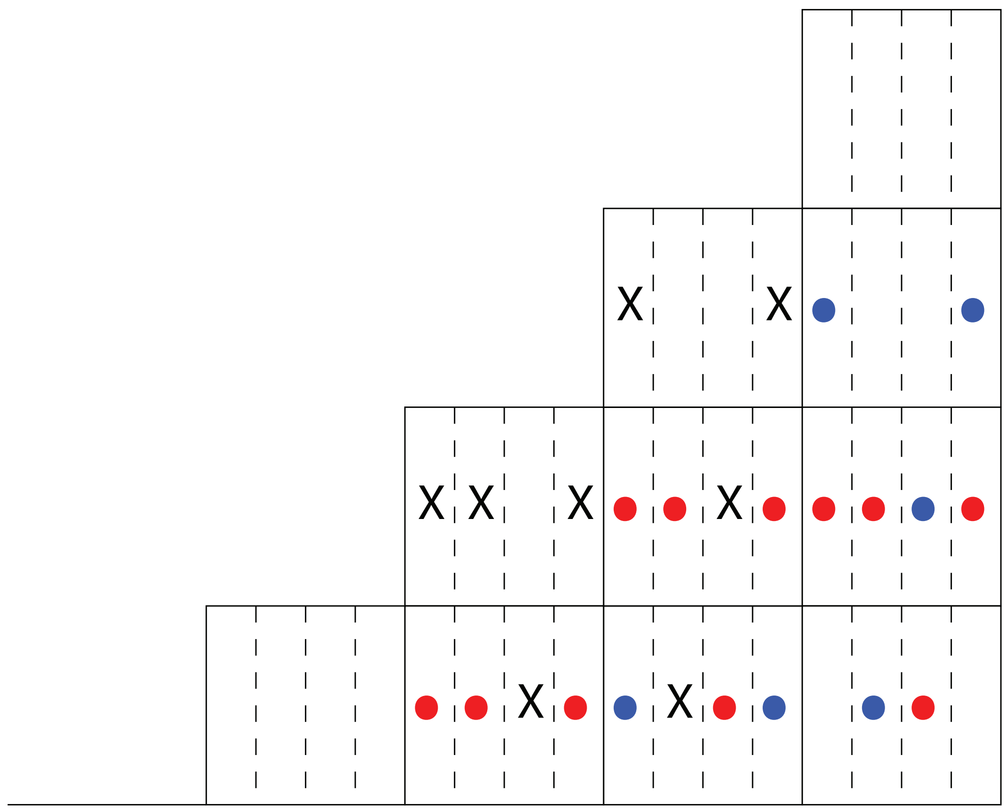

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Rook Numbers

Definition: The rook number $r(n, k, \mathcal{B}_n^m)$ is the number of ways of placing $m(n-k)$ attacking rooks on the board $\mathcal{B}_n^m$.

We notice that $r(n+1, k, \mathcal{B}_{n+1}^m) = r(n, k-1, \mathcal{B}_n^m) + k^m r(n, k, \mathcal{B}_n^m)$, and we can define $r(0, 0, \mathcal{B}_0^m) = 1$. Moreover, $r(n, k, \mathcal{B}_n^m) = 0$ if $n < k \leq 0$. This is the same recursion as $S(n, k, x^m)$, that is, $S(n, k, x^m) = r(n, k, \mathcal{B}_n^m)$.

Note: Similar definitions could give rook theoretic interpretations for $c(n, k, x^m)$ and $s(n, k, x^m)$, and all of these can be extended for appropriate $p(x) = \sum_{i=1}^{s} a_i x^i$.

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Poly-Stirling Numbers

Generalizations of Known Formulas

From these rook theoretical interpretations we can get purely combinatorial derivations of the following formulas:

- $$\sum_{n\geq k} S(n,k,p)x^n = \frac{x^k}{(1-p(1)x)(1-p(2)x)\cdots(1-p(k)x)}.$$
- $$\sum_{k=0}^{n}\sum_{j=0}^{k} S(n,k,p)s(k,j,p) = \chi(n=j).$$
- $$(p(x))^n = \sum_{k=0}^{n} S(n,k,p)(p(x))(p(x)-p(1))\cdots(p(x)-p(k-1)).$$

Poly-Stirling Numbers

Set Partition Interpretations

Like with regular Stirling numbers, we can get set partition interpretations of the numbers $S(n, k, p(x))$.

- $S(n, k, x^m)$ is the number of m-tuples of partitions of $\{1, 2, \ldots, n\}$ in to k parts such that the set of minimal elements of the parts for each partition is the same.
- Example: $S(4, 2, x^2) = 21$

$$\{1\}\{2,3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}, \{1,3,4\}\{2\}$$

$$\{1,2\}\{3,4\}, \{1,2,4\}\{3\}$$

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An Open Problem

There are many formulas which generalize quite nicely, but also many that do not.

For example, we would like to find an exponential generating function for $S(n, p, k)$. This can be done for $p(x) = x^2$, and maybe for $p(x) = x^m$, but it does not seem possible for arbitrary $p(x)$.

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The End

Thanks for listening, and feel free to ask questions.

(I will post these slides to my web page under "Research" later this week.)

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