How Is Mathematical Research Done?

Brian Miceli Trinity University Majors' Seminar August 26th, 2010



A Quick Outline

- Some Definitions
- What is the problem?
- How was it solved?

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Example: Given the alphabet $\{A, B, C, \ldots, Y, Z\}$, a mathematician considers ZHXAUBBSRVE to be a word of length 11. XAUB is a subword of $ZH\underline{XAUB}BSRVE$ of length 4.

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More Definitions

For this talk we will use the alphabet $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$

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Definition: A word, w, contains another word, u, if there exists a subword of w, v, such that each letter of v is greater than or equal to its corresponding letter in u. If w does not contain u, then we say that w avoids u.

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A Research Question

With just those few definitions, we are now in a position to look at the following research question, which was posed to me in July by a colleague in San Diego.

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We want words which contain u = 13. Since u has length 2 and weight 4, the words we are interested in must have length at least 2 and a weight of at least 4.

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Example: (n = 6) There are 12 words - 15, 24, 1<u>14</u>, <u>14</u>1, 33, 1<u>23</u>, <u>13</u>2, 2<u>13</u>, <u>23</u>1, 11<u>13</u>, <u>113</u>1, <u>13</u>11.



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Example: (n = 7, 8) There are 30 words of weight 7 and 68 of weight 8.



Now We Have Some Data

We have a sequence of terms that looks like:

 $0, 0, 0, 1, 4, 12, 30, 68, \ldots$

We now have to ask ourselves if this sequence tells us anything. A good place to start is a web site called The On-Line Encyclopedia of Integer Sequences, or OIES...



What Did We Find?

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This raises the question: "What do binary words have to do with our words?"



A Bit on Binary Words

- A binary word is any word from the alphabet $\{0, 1\}$.
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Example: There are $2^3 = 8$ binary words of length 3:

000,001,010,100,011,101,110,111.



What to Do Next?

How many words of weight n contain exactly one occurrence of the word 13?

At this point we got stuck trying to figure out how binary words were related to words from the alphabet $\mathcal{A} = \{1, 2, \dots, 9\}$.



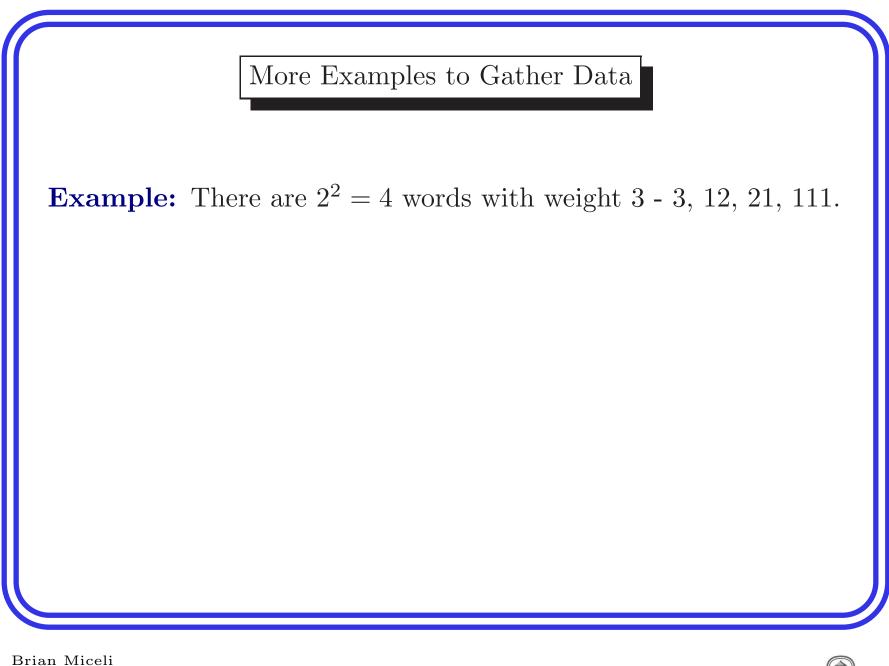
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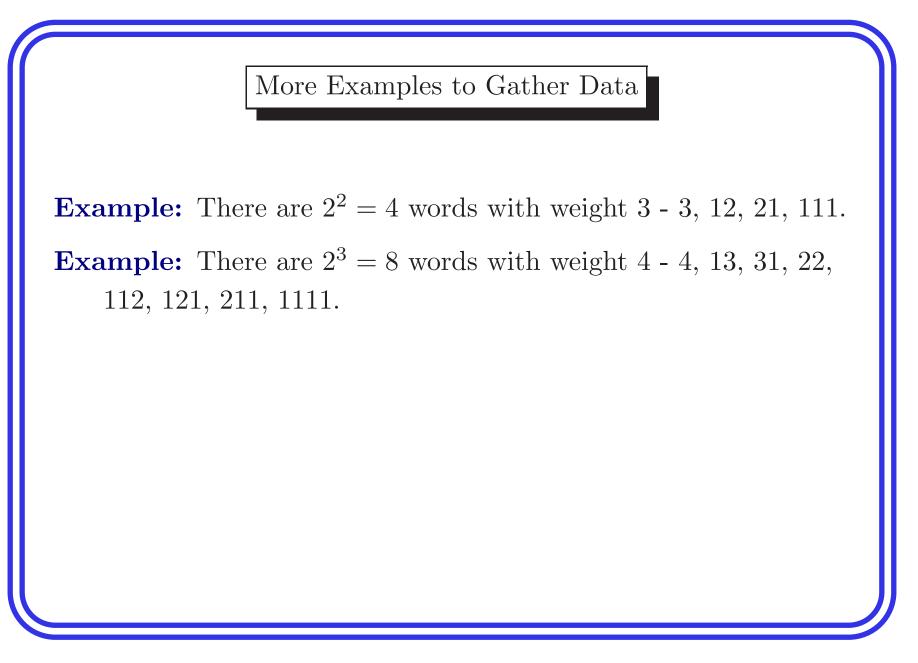
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One attack was to write down all of the words of weight n for various values of n to see if we could figure out some similarity between all words that avoid 13, had exactly one occurrence of 13, etc..

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More Examples to Gather Data

Example: There are $2^2 = 4$ words with weight 3 - 3, 12, 21, 111.

Example: There are $2^3 = 8$ words with weight 4 - 4, 13, 31, 22, 112, 121, 211, 1111.

Example: There are 2⁴ = 16 words with weight 5 - 5, 14, 41, 23, 32, 113, 131, 311, 122, 212, 221, 1112, 1121, 1211, 2111, 11111.



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Example: There are $2^5 = 32$ words with weight 6 and $2^6 = 64$ words with weight 7.



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Example: There are $2^5 = 32$ words with weight 6 and $2^6 = 64$ words with weight 7.

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Theorem: The number of binary words of length n is equal to the number of words from \mathcal{A} with weight n + 1.

The question now becomes, can we prove this theorem?



Making the Connection

YES!

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YES!

We will illustrate this with an example.





Suppose we are given a binary word of length n = 8, say

01100010.



Making the Connection

Suppose we are given a binary word of length n = 8, say

01100010.

We wish to turn this into a word with weight n + 1 = 9. So, consider a string of 9 1's:

1 1 1 1 1 1 1 1 1.



Making the Connection

$1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1.$

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1 1 1 1 1 1 1 1 1.

We see that there are 8 spaces between these 9 1's, and so we can place each letter from our binary word in one of those spaces, in order:



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We now replace every one of the subscripted 0's with a "+" and the subscripted 1's with a break,







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$1+1 \mid 1 \mid 1+1+1+1 \mid 1+1,$







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We now replace every one of the subscripted 0's with a "+" and the subscripted 1's with a break,

1+1 | 1 | 1+1+1+1 | 1+1,

and we turn this into the word 2142. Notice that this word has weight 2 + 1 + 4 + 2 = 9, as desired.



Making the Connection

The next question we have to ask is can we go the other way? That is, given a word from \mathcal{A} with weight n + 1, can we match it with a binary word of length n?



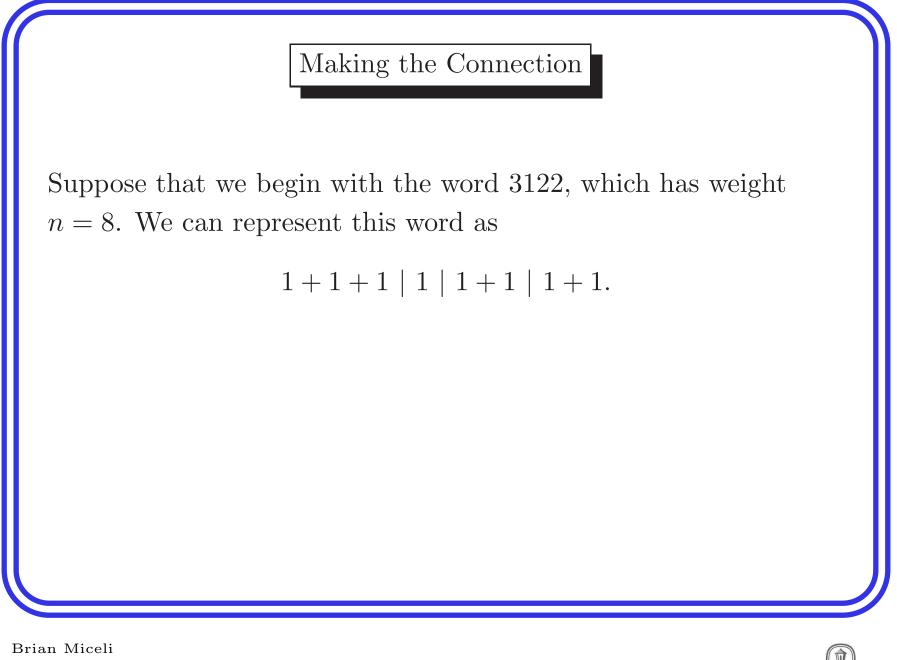
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The next question we have to ask is can we go the other way? That is, given a word from \mathcal{A} with weight n + 1, can we match it with a binary word of length n?

The answer is "yes," and again we illustrate this by example.







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Suppose that we begin with the word 3122, which has weight n = 8. We can represent this word as

$1 + 1 + 1 \mid 1 \mid 1 + 1 \mid 1 + 1$.

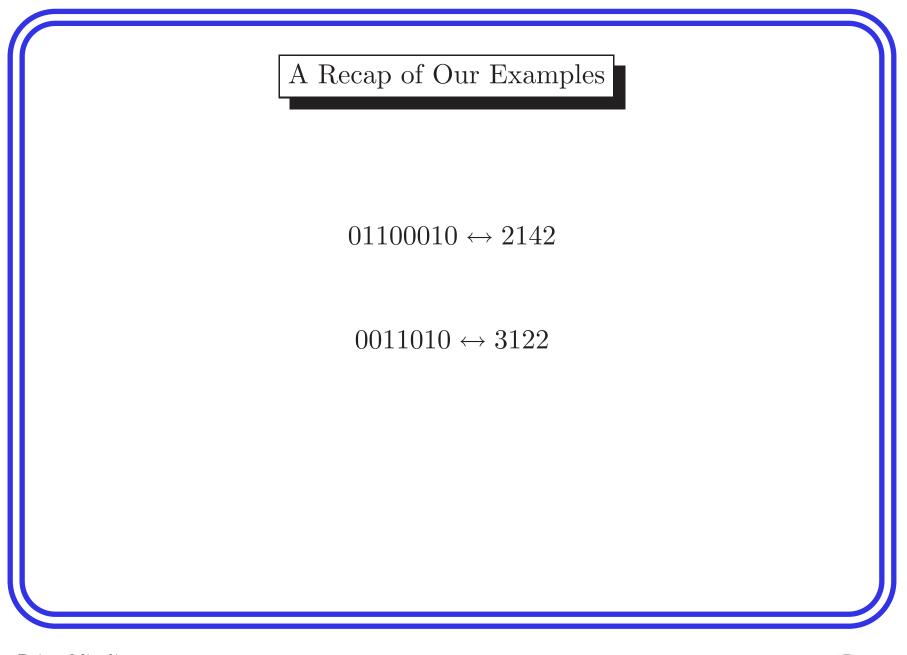
We now replace the "+" signs with 0's and the breaks with 1's to get

```
1_0 1_0 1_1 1_1 1_0 1_1 1_0 1_1
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from which we obtain a binary word of length n - 1 = 7:

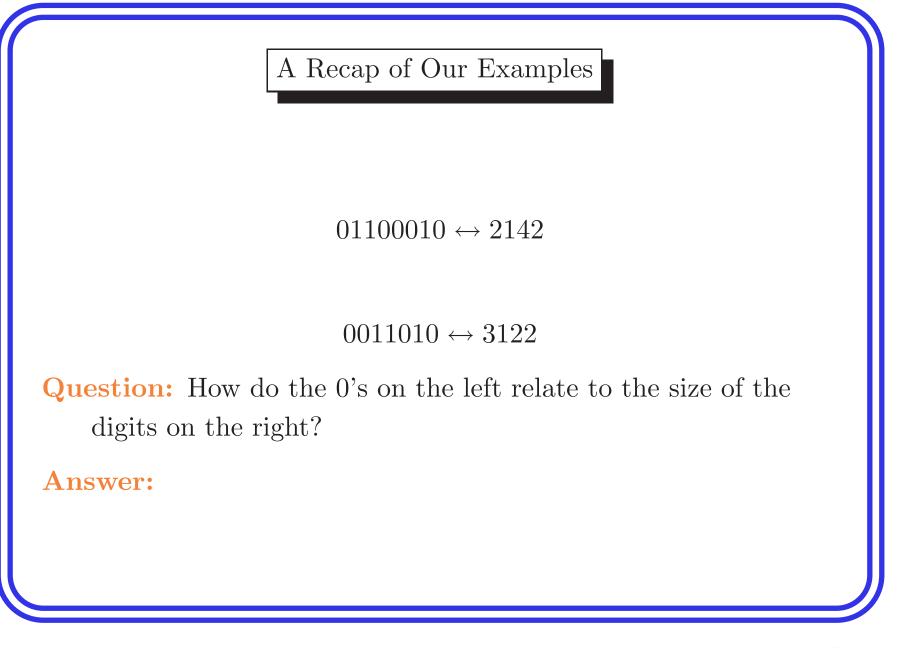
0011010.

















$01100010\leftrightarrow 2142$

$0011010\leftrightarrow 3122$

Question: How do the 0's on the left relate to the size of the digits on the right?

Answer: It seems like if we have k consecutive 0's, then they correspond to a digit of size k + 1. For example, 100 would correspond to the word 13.





Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a word of weight n + 1 that contains exactly one occurrence of the subword 13.



A Theorem

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Recall that the OEIS said that this sequence is the number of binary words that contain exactly one occurrence of the binary subword 001, which is not quite what we have.





Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a binary word that contains exactly one occurrence of the string 001.



The Final Piece

Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a binary word that contains exactly one occurrence of the string 001.

Proof: Suppose a binary word, b, contains exactly one occurrence of the string 100. Then the binary word, b_r , made by reversing the order of b, has the same length as b, and moreover, it contains exactly one occurrence of the string 001.



A More General Result

By putting all these results together, we can actually obtain a more general result, given here.

Theorem: For every binary word of length n which contains exactly j occurrences of the string 100, there is a word from the alphabet \mathcal{A} with weight n + 1 that contains exactly joccurrences of the word 13.





Thanks for listening, and feel free to ask questions.

