How Is Mathematical Research Done?

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Majors’ Seminar
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Mathematical Research

A Quick Outline

• Some Definitions

• What is the problem?

• How was it solved?
Definitions: An alphabet is a set of characters, and the elements of that set are called letters.
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**Definitions:** An *alphabet* is a set of characters, and the elements of that set are called *letters*. A *word* is any string of letters from a specified alphabet, and a *subword* is any consecutive set of letters that appear within another word. The *length* of a word is the number of letters that comprise that word.

**Example:** Given the alphabet \{A, B, C, \ldots, Y, Z\}, a mathematician considers `ZHXAUBBSRVE` to be a word of length 11. `XAUB` is a subword of `ZHXAUBBSRVE` of length 4.
For this talk we will use the alphabet $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

**Definition:** The *weight* of a word with letters from $\mathcal{A}$ is the sum of the letters of that word.
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**Definition:** A word, $w$, *contains* another word, $u$, if there exists a subword of $w$, $v$, such that each letter of $v$ is greater than or equal to its corresponding letter in $u$. If $w$ does not contain $u$, then we say that $w$ *avoids* $u$. 

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Example: Let $w = 81425336$. Then the word $w$ contains two occurrences of the word $y = 15$: 81425336.
Example: Let \( w = 81425336 \). Then the word \( w \) contains two occurrences of the word \( y = 15: 8142\underline{5}336 \). However, \( w \) avoids the word \( z = 45 \) since there are no consecutive letters of \( w \) where the first letter is greater than or equal to 4 and then next is greater than or equal to 5.

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*How many words of weight $n$ contain exactly one occurrence of the word 13?*
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How many words of weight \( n \) contain exactly one occurrence of the word 13?

We want words which contain \( u = 13 \). Since \( u \) has length 2 and weight 4, the words we are interested in must have length at least 2 and a weight of at least 4.
We first generate examples of words with exactly one containment of 13.

**Example:** There are 0 words of weight $n = 1, 2, 3$. 
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Examples to Gather Data

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Example: \((n = 6)\) There are 12 words - 15, 24, 114, 141, 33, 123, 132, 213, 231, 1113, 1131, 1311.
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**Example:** \((n = 6)\) There are 12 words - 15, 24, 114, 141, 33, 123, 132, 213, 231, 1113, 1131, 1311.

**Example:** \((n = 7, 8)\) There are 30 words of weight 7 and 68 of weight 8.
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Now We Have Some Data

We have a sequence of terms that looks like:

\[0, 0, 0, 1, 4, 12, 30, 68, \ldots\]

We now have to ask ourselves if this sequence tells us anything. A good place to start is a web site called The On-Line Encyclopedia of Integer Sequences, or OIES…

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What Did We Find?

The OEIS says that this sequence may be the number of binary words that contain exactly one occurrence of the binary subword 001.
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This raises the question: “What do binary words have to do with our words?”
A Bit on Binary Words

- A binary word is any word from the alphabet \{0, 1\}.
- There are $2^n$ binary words of length $n$. 
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**Example:** There are \(2^3 = 8\) binary words of length 3:

000, 001, 010, 100, 011, 101, 110, 111.
What to Do Next?

How many words of weight $n$ contain exactly one occurrence of the word 13?

At this point we got stuck trying to figure out how binary words were related to words from the alphabet $\mathcal{A} = \{1, 2, \ldots, 9\}$.
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At this point we got stuck trying to figure out how binary words were related to words from the alphabet $A = \{1, 2, \ldots, 9\}$.

One attack was to write down all of the words of weight $n$ for various values of $n$ to see if we could figure out some similarity between all words that avoid 13, had exactly one occurrence of 13, etc..
More Examples to Gather Data

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**Example:** There are $2^3 = 8$ words with weight 4 - 4, 13, 31, 22, 112, 121, 211, 1111.

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Example: There are $2^2 = 4$ words with weight 3 - 3, 12, 21, 111.

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Example: There are $2^4 = 16$ words with weight 5 - 5, 14, 41, 23, 32, 113, 131, 311, 122, 212, 221, 1112, 1121, 1211, 2111, 11111.
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**Example:** There are $2^3 = 8$ words with weight 4 - 4, 13, 31, 22, 112, 121, 211, 1111.

**Example:** There are $2^4 = 16$ words with weight 5 - 5, 14, 41, 23, 32, 113, 131, 311, 122, 212, 221, 1112, 1121, 1211, 2111, 11111.

**Example:** There are $2^5 = 32$ words with weight 6 and $2^6 = 64$ words with weight 7.

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**Example:** There are $2^5 = 32$ words with weight 6 and $2^6 = 64$ words with weight 7.

**Conjecture:** The number of words from $\mathcal{A}$ of weight $n + 1$ is $2^n$. 
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If this conjecture is correct, then we have the following theorem.

**Theorem:** The number of binary words of length $n$ is equal to the number of words from $\mathcal{A}$ with weight $n + 1$.

The question now becomes, can we prove this theorem?
Making the Connection

YES!
We will illustrate this with an example.
Suppose we are given a binary word of length \( n = 8 \), say

\[ 01100010. \]
Suppose we are given a binary word of length $n = 8$, say

$$01100010.$$  

We wish to turn this into a word with weight $n + 1 = 9$. So, consider a string of 9 1’s:

$$1 1 1 1 1 1 1 1 1.$$
We see that there are 8 spaces between these 9 1's, and so we can place each letter from our binary word in one of those spaces, in order:

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We now replace every one of the subscripted 0’s with a “+” and the subscripted 1’s with a break,
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\[ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1. \]

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\[ 1 + 1 \mid 1 \mid 1 + 1 + 1 + 1 \mid 1 + 1, \]
We now replace every one of the subscripted 0’s with a “+” and the subscripted 1’s with a break,

1 + 1 | 1 | 1 + 1 + 1 + 1 | 1 + 1,

and we turn this into the word 2142. Notice that this word has weight 2 + 1 + 4 + 2 = 9, as desired.
The next question we have to ask is can we go the other way? That is, given a word from $\mathcal{A}$ with weight $n + 1$, can we match it with a binary word of length $n$?
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The answer is “yes,” and again we illustrate this by example.
Suppose that we begin with the word 3122, which has weight \( n = 8 \). We can represent this word as

\[
1 + 1 + 1 \mid 1 \mid 1 + 1 \mid 1 + 1.
\]
Suppose that we begin with the word 3122, which has weight $n = 8$. We can represent this word as

$$1 + 1 + 1 \mid 1 \mid 1 + 1 \mid 1 + 1.$$ 

We now replace the “+” signs with 0’s and the breaks with 1’s to get

$$1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1,$$

from which we obtain a binary word of length $n - 1 = 7$:

$$0011010.$$
A Recap of Our Examples

01100010 ↔ 2142

0011010 ↔ 3122
A Recap of Our Examples

\[01100010 \leftrightarrow 2142\]

\[0011010 \leftrightarrow 3122\]

**Question:** How do the 0’s on the left relate to the size of the digits on the right?

**Answer:**
A Recap of Our Examples

01100010 $\leftrightarrow$ 2142

0011010 $\leftrightarrow$ 3122

**Question:** How do the 0’s on the left relate to the size of the digits on the right?

**Answer:** It seems like if we have $k$ consecutive 0’s, then they correspond to a digit of size $k + 1$. For example, 100 would correspond to the word 13.
A Theorem

**Theorem:** For every binary word of length $n$ which contains exactly one occurrence of the string 100, there is a word of weight $n + 1$ that contains exactly one occurrence of the subword 13.
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**Theorem:** For every binary word of length $n$ which contains exactly one occurrence of the string 100, there is a word of weight $n + 1$ that contains exactly one occurrence of the subword 13.

Recall that the OEIS said that this sequence is the number of binary words that contain exactly one occurrence of the binary subword 001, which is not quite what we have.
The Theorem: For every binary word of length $n$ which contains exactly one occurrence of the string 100, there is a binary word that contains exactly one occurrence of the string 001.

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Theorem: For every binary word of length \( n \) which contains exactly one occurrence of the string 100, there is a binary word that contains exactly one occurrence of the string 001.

Proof: Suppose a binary word, \( b \), contains exactly one occurrence of the string 100. Then the binary word, \( b_r \), made by reversing the order of \( b \), has the same length as \( b \), and moreover, it contains exactly one occurrence of the string 001.
By putting all these results together, we can actually obtain a more general result, given here.

**Theorem:** For every binary word of length $n$ which contains exactly $j$ occurrences of the string 100, there is a word from the alphabet $A$ with weight $n + 1$ that contains exactly $j$ occurrences of the word 13.
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The End

Thanks for listening, and feel free to ask questions.

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