How Is Mathematical Research Done?

Brian Miceli
Trinity University
Majors' Seminar
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A Quick Outline

- Some Definitions
- What is the problem?
- How was it solved?

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Definitions

Definitions: An *alphabet* is a set of characters, and the elements of that set are called *letters*.

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Example: Given the alphabet $\{A, B, C, \ldots, Y, Z\}$, a mathematician considers $ZHXAUBBSRVE$ to be a word of length 11. $XAUB$ is a subword of $ZH\underline{XAUB}BSRVE$ of length 4.

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More Definitions

For this talk we will use the alphabet $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Definition: The *weight* of a word with letters from $\mathcal{A}$ is the sum of the letters of that word.

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Example: Since 4+1+5+9+2=21, the word 41592 has weight 21.

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Definition: A word, w, *contains* another word, u, if there exists a subword of w, v, such that each letter of v is greater than or equal to its corresponding letter in u. If w does not contain u, then we say that w *avoids* u.

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An Example

Example: Let $w = 81425336$. Then the word w contains two occurrences of the word $y = 15$: $814\underline{25}3\underline{36}$.

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An Example

Example: Let $w = 81425336$. Then the word w contains two occurrences of the word $y = 15$: $814\underline{25}3\underline{36}$. However, w avoids the word $z = 45$ since there are no consecutive letters of w where the first letter is greater than or equal to 4 and then next is greater than or equal to 5.

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A Research Question

With just those few definitions, we are now in a position to look at the following research question, which was posed to me in July by a colleague in San Diego.

How many words of weight n contain
exactly one occurrence of the word 13?

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We want words which contain $u = 13$. Since u has length 2 and weight 4, the words we are interested in must have length at least 2 and a weight of at least 4.

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Examples to Gather Data

We first generate examples of words with exactly one containment of 13.

Example: There are 0 words of weight $n = 1, 2, 3$.

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Example: $(n = 4)$ There is 1 word - 13.

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Example: There are 0 words of weight $n = 1, 2, 3$.

Example: $(n = 4)$ There is 1 word - 13.

Example: $(n = 5)$ There are 4 words - 14, 23, $1\underline{13}$, $\underline{13}1$.

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Example: $(n = 5)$ There are 4 words - 14, 23, $1\underline{13}$, $\underline{13}1$.

Example: $(n = 6)$ There are 12 words - 15, 24, $1\underline{14}$, $\underline{14}1$, 33, $1\underline{23}$, $\underline{13}2$, $2\underline{13}$, $\underline{23}1$, $11\underline{13}$, $1\underline{13}1$, $\underline{13}11$.

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Example: $(n = 7, 8)$ There are 30 words of weight 7 and 68 of weight 8.

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Now We Have Some Data

We have a sequence of terms that looks like:

$$0, 0, 0, 1, 4, 12, 30, 68, \ldots.$$

We now have to ask ourselves if this sequence tells us anything. A good place to start is a web site called The On-Line Encyclopedia of Integer Sequences, or OIES...

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What Did We Find?

The OEIS says that this sequence may be the number of binary words that contain exactly one occurrence of the binary subword 001.

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What Did We Find?

The OEIS says that this sequence may be the number of binary words that contain exactly one occurrence of the binary subword 001.

This raises the question: "What do binary words have to do with our words?"

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A Bit on Binary Words

- A binary word is any word from the alphabet $\{0, 1\}$.
- There are 2^n binary words of length n.

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Example: There are $2^3 = 8$ binary words of length 3:

$$000, 001, 010, 100, 011, 101, 110, 111.$$

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What to Do Next?

How many words of weight n contain exactly one occurrence of the word 13?

At this point we got stuck trying to figure out how binary words were related to words from the alphabet $\mathcal{A} = \{1, 2, \ldots, 9\}$.

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One attack was to write down all of the words of weight n for various values of n to see if we could figure out some similarity between all words that avoid 13, had exactly one occurrence of 13, etc..

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More Examples to Gather Data

Example: There are $2^2 = 4$ words with weight 3 - 3, 12, 21, 111.

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More Examples to Gather Data

Example: There are $2^2 = 4$ words with weight 3 - 3, 12, 21, 111.

Example: There are $2^3 = 8$ words with weight 4 - 4, 13, 31, 22, 112, 121, 211, 1111.

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Example: There are $2^2 = 4$ words with weight 3 - 3, 12, 21, 111.

Example: There are $2^3 = 8$ words with weight 4 - 4, 13, 31, 22, 112, 121, 211, 1111.

Example: There are $2^4 = 16$ words with weight 5 - 5, 14, 41, 23, 32, 113, 131, 311, 122, 212, 221, 1112, 1121, 1211, 2111, 11111.

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More Examples to Gather Data

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Example: There are $2^4 = 16$ words with weight 5 - 5, 14, 41, 23, 32, 113, 131, 311, 122, 212, 221, 1112, 1121, 1211, 2111, 11111.

Example: There are $2^5 = 32$ words with weight 6 and $2^6 = 64$ words with weight 7.

More Examples to Gather Data

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Example: There are $2^3 = 8$ words with weight 4 - 4, 13, 31, 22, 112, 121, 211, 1111.

Example: There are $2^4 = 16$ words with weight 5 - 5, 14, 41, 23, 32, 113, 131, 311, 122, 212, 221, 1112, 1121, 1211, 2111, 11111.

Example: There are $2^5 = 32$ words with weight 6 and $2^6 = 64$ words with weight 7.

Conjecture: The number of words from $\mathcal{A}$ of weight $n+1$ is 2^n.

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Making the Connection

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If this conjecture is correct, then we have the following theorem.

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Making the Connection

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Theorem: The number of binary words of length n is equal to the number of words from $\mathcal{A}$ with weight $n+1$.

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The question now becomes, can we prove this theorem?

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Making the Connection

YES!

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Making the Connection

YES!

We will illustrate this with an example.

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Making the Connection

Suppose we are given a binary word of length $n = 8$, say

$$01100010.$$

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Making the Connection

Suppose we are given a binary word of length $n = 8$, say

$$01100010.$$

We wish to turn this into a word with weight $n + 1 = 9$. So, consider a string of 9 1's:

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1.$$

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Making the Connection

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$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1.$$

We see that there are 8 spaces between these 9 1's, and so we can place each letter from our binary word in one of those spaces, in order:

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$$1_{\,0}1_{\,1}1_{\,1}1_{\,0}1_{\,0}1_{\,0}1_{\,1}1_{\,0}1.$$

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Making the Connection

$$1_{\,0}\,1_{\,1}\,1_{\,1}\,1_{\,0}\,1_{\,0}\,1_{\,0}\,1_{\,1}\,1_{\,0}\,1.$$

We now replace every one of the subscripted 0's with a "+" and the subscripted 1's with a break,

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$$1\ _0\ 1\ _1\ 1\ _1\ 1\ _0\ 1\ _0\ 1\ _0\ 1\ _1\ 1\ _0\ 1.$$

We now replace every one of the subscripted 0's with a "+" and the subscripted 1's with a break,

$$1+1 \mid 1 \mid 1+1+1+1 \mid 1+1,$$

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$$1_{\;0}\,1_{\;1}\,1_{\;1}\,1_{\;0}\,1_{\;0}\,1_{\;0}\,1_{\;1}\,1_{\;0}\,1.$$

We now replace every one of the subscripted 0's with a "+" and the subscripted 1's with a break,

$$1+1\mid 1\mid 1+1+1+1\mid 1+1,$$

and we turn this into the word 2142. Notice that this word has weight $2+1+4+2=9$, as desired.

Making the Connection

The next question we have to ask is can we go the other way? That is, given a word from $\mathcal{A}$ with weight $n+1$, can we match it with a binary word of length n?

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Making the Connection

The next question we have to ask is can we go the other way? That is, given a word from $\mathcal{A}$ with weight $n+1$, can we match it with a binary word of length n?

The answer is “yes,” and again we illustrate this by example.

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Making the Connection

Suppose that we begin with the word 3122, which has weight $n = 8$. We can represent this word as

$$1+1+1 \mid 1 \mid 1+1 \mid 1+1.$$

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Making the Connection

Suppose that we begin with the word 3122, which has weight $n = 8$. We can represent this word as

$$1 + 1 + 1 \mid 1 \mid 1 + 1 \mid 1 + 1.$$

We now replace the "+" signs with 0's and the breaks with 1's to get

$$1 \;{}_0\; 1 \;{}_0\; 1 \;{}_1\; 1 \;{}_1\; 1 \;{}_0\; 1 \;{}_1\; 1 \;{}_0\; 1,$$

from which we obtain a binary word of length $n - 1 = 7$:

$$0011010.$$

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A Recap of Our Examples

$$01100010 \leftrightarrow 2142$$

$$0011010 \leftrightarrow 3122$$

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$$01100010 \leftrightarrow 2142$$

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Question: How do the 0's on the left relate to the size of the digits on the right?

Answer:

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A Recap of Our Examples

$$01100010 \leftrightarrow 2142$$

$$0011010 \leftrightarrow 3122$$

Question: How do the 0's on the left relate to the size of the digits on the right?

Answer: It seems like if we have k consecutive 0's, then they correspond to a digit of size $k+1$. For example, 100 would correspond to the word 13.

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A Theorem

Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a word of weight $n+1$ that contains exactly one occurrence of the subword 13.

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A Theorem

Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a word of weight $n+1$ that contains exactly one occurrence of the subword 13.

Recall that the OEIS said that this sequence is the number of binary words that contain exactly one occurrence of the binary subword 001, which is not quite what we have.

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The Final Piece

Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a binary word that contains exactly one occurrence of the string 001.

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The Final Piece

Theorem: For every binary word of length n which contains exactly one occurrence of the string 100, there is a binary word that contains exactly one occurrence of the string 001.

Proof: Suppose a binary word, b, contains exactly one occurrence of the string 100. Then the binary word, b_r, made by reversing the order of b, has the same length as b, and moreover, it contains exactly one occurrence of the string 001.

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A More General Result

By putting all these results together, we can actually obtain a more general result, given here.

Theorem: For every binary word of length n which contains exactly j occurrences of the string 100, there is a word from the alphabet $\mathcal{A}$ with weight $n+1$ that contains exactly j occurrences of the word 13.

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The End

Thanks for listening, and feel free to ask questions.

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