Generalized Interval Embeddings

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Outline

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  - "Rationality, irrationality, and Wilf equivalence in g.f.o.," *EJC* (2009) by Kitaev, Liese, Remmel, and Sagan.
  - "Generating functions for Wilf equivalence under the g.f.o.," *JIS* (2011) by Langley, Liese, and Remmel

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We say that $u \in \mathbb{N}^*$ is a *factor* of $v \in \mathbb{N}^*$ if there exist $w_1, w_2 \in \mathbb{N}^*$ such that $v = w_1 u w_2$. If $w_1 = \epsilon$ ($w_2 = \epsilon$), then we say that u is a prefix (suffix) of v.

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Word Statistics

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We then define the *weight of u* to be $wt(u) = x^{\Sigma u} t^{\ell(u)}$.

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Given any poset $P = (\mathbb{N}, \leq_P)$ and two words $u, w \in \mathbb{N}^*$, we say that there is an *embedding u into w* if there exists a factor $z = z_1 z_2 \cdots z_k$ of w such that for every $1 \leq i \leq k$, $u_i \leq_P z_i$.

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Define $\mathcal{S}^P(u)$ to be the set of all words w such that the only embedding of u into w occurs at the right end of w, that is, the embedding of u occurs in a suffix of w.

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Set $\mathcal{S}^P(u, x, t) = \sum_{w \in \mathcal{S}^P(u)} wt(w)$. We say that words u and v are *P-Wilf Equivalent*, denoted by $u \sim_P v$, if

$$\mathcal{S}^P(u, x, t) = \mathcal{S}^P(v, x, t).$$

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- The functions $\mathcal{E}^P, \mathcal{A}^P, \mathcal{S}^P$ are rational for any choice of u.

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- $u \sim u^r$

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- $u \sim u^r$, and if $u \sim v$, then $1u \sim 1v$ and $u^+ \sim v^+$.

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- The functions $\mathcal{E}^P, \mathcal{A}^P, \mathcal{S}^P$ are rational for any choice of u.
- $u \sim u^r$, and if $u \sim v$, then $1u \sim 1v$ and $u^+ \sim v^+$.
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- $u \sim u^r$, and if $u \sim v$, then $1u \sim 1v$ and $u^+ \sim v^+$.
- Is it the case that $u \sim v$ implies u, v are rearrangements of one another? (The converse is definitely not true.)

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1. $u_i \leq z_i$, and
2. $u_i \equiv z_i \mod m$.

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Define $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{m_1,n_1}, \mathcal{I}^{\mathcal{P}}_{m_2,n_2}, \ldots, \mathcal{I}^{\mathcal{P}}_{m_k,n_k}\}$, where for each $1 \leq i \leq k$, $m_i \leq_{\mathcal{P}} n_i$ with either $m_i, n_i \in \mathbb{N}$ or $m_i \in \mathbb{N}$ and $n_i = \infty$.

Definitions

We say that w contains an *interval-embedding of* $\vec{U}$ *relative to* $\mathcal{P}$ if there is a factor z of w such that for every $1 \leq i \leq k$, $z_i \in \mathcal{I}^{\mathcal{P}}_{m_i,n_i}$.

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Finally, we define

$$\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t) = \sum_{w \in \mathcal{S}^{\mathcal{P}}(\vec{U})} wt(w).$$

Relation to Previous Models

We say that w contains an *interval-embedding of* $\vec{U}$ *relative to* $\mathcal{P}$ if there is a factor z of w such that for every $1 \leq i \leq k$, $z_i \in \mathcal{I}^{\mathcal{P}}_{m_i,n_i}$.

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If $\mathcal{P} = \mathcal{P}_m = (\mathbb{N}, \leq_m)$ and $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{m_1,n_1}, \mathcal{I}^{\mathcal{P}}_{m_2,n_2}, \ldots, \mathcal{I}^{\mathcal{P}}_{m_k,n_k}\}$ with $n_i = \infty$ for all i, then this is the L.L.R. version of (modular) embedding the word $u = m_1 m_2 \cdots m_k$.

Interval-Wilf Equivalence

We say that $\vec{U}$ and $\vec{V}$ are *interval-Wilf equivalent* with respect to $\mathcal{P}$, denoted at $\vec{U} \sim_{\mathcal{P}} \vec{V}$, if

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As an example, suppose $\mathcal{P} = (\mathbb{N}, \leq)$, $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{3,8}, \mathcal{I}^{\mathcal{P}}_{1,8}, \mathcal{I}^{\mathcal{P}}_{2,8}\}$, and $\vec{U}^r = \{\mathcal{I}^{\mathcal{P}}_{2,8}, \mathcal{I}^{\mathcal{P}}_{1,8}, \mathcal{I}^{\mathcal{P}}_{3,8}\}$.

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Theorem

If $\mathcal{P} = (\mathbb{N}, \leq)$, then $\vec{U} \sim_{\mathcal{P}} \vec{U}^r$.

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Rationality of Generating Functions

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If $\mathcal{P} = (\mathbb{N}, \leq)$, then the functions $\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t)$, $\mathcal{E}^{\mathcal{P}}(\vec{U}, x, t)$, and $\mathcal{A}^{\mathcal{P}}(\vec{U}, x, t)$ are all rational for any $\vec{U}$.

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Proof. This follows from the K.L.R.S. results.

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Rearrangment

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Let $\mathcal{P} = (\mathbb{N}, \leq)$ *and fix* $n \in \mathbb{N}$. *If* $\vec{U} \sim_{\mathcal{P}} \vec{V}$ *with* $\vec{U} = \{\mathcal{I}^{\mathcal{P}}_{m_1,n}, \mathcal{I}^{\mathcal{P}}_{m_2,n}, \ldots, \mathcal{I}^{\mathcal{P}}_{m_k,n}\}$ *and* $\vec{V} = \{\mathcal{I}^{\mathcal{P}}_{r_1,n}, \mathcal{I}^{\mathcal{P}}_{r_2,n}, \ldots, \mathcal{I}^{\mathcal{P}}_{r_\ell,n}\}$, *then* $\vec{U}$ *and* $\vec{V}$ *are rearrangements of one another.*

Proof.

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Proof. Let $u = m_1 m_2 \cdots m_k$ and $v = r_1 r_2 \cdots r_\ell$.

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Proof. Let $u = m_1 m_2 \cdots m_k$ and $v = r_1 r_2 \cdots r_\ell$. We first note that we must have $k = \ell$ and $\Sigma u = \Sigma v$.

Now, suppose $\vec{U}$ and $\vec{V}$ are not rearrangements of each other.

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Proof. (cont...)

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Proof. (cont...) Let $\bar{r}$ be as described.

Consider the coefficient of $x^{\Sigma u+n-\bar{r}+1}t^k$ in $\mathcal{S}^{\mathcal{P}}(\vec{U}, x, t)$ and $\mathcal{S}^{\mathcal{P}}(\vec{V}, x, t)$.

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Proof. (cont...) Let $\overline{r}$ be as described.

Consider the coefficient of $x^{\Sigma u+n-\overline{r}+1}t^k$ in $\mathcal{S}^{\mathcal{P}}(\vec{U},x,t)$ and $\mathcal{S}^{\mathcal{P}}(\vec{V},x,t)$.

These are not equal, a contradiction.

Generating Functions
Some Preliminary Results

Suppose $\mathcal{P} = (\mathbb{N}, \leq)$.

Theorem

Suppose $i, j \in \mathbb{N}$ with $i \leq j$. Then

$$\mathcal{S}^{\mathcal{P}}(\{[i,j]\}, t, x) = \frac{tx^i(1 - x^{j-i-1})}{1 - x - tx(1 - x^{i-1}) - tx^{j+1}}.$$

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Theorem

Suppose $i, j \in \mathbb{N}$ with $1 < i \leq j$. Then

$$\mathcal{S}^{\mathcal{P}}(\{[1,1],[i,j]\}, t, x) = \frac{t^2x^{i+1}(1 - x^{j-i-1})}{1 - x - tx(1 - x^{i-1}) - t^2x^{j+2}}.$$

Future Work

- Look at Interval-Wilf Equivalence classes.

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- It would still be nice to push these results up to get the general rearrangement conjecture.

Thank you.