On Rook & Stirling Numbers

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Trinity University
Discrete Math Seminar
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A Quick Outline

- What is a *Stirling number*?
- What is a *rook number*?
- How are they related?
- A generalization of the Stirling numbers with some combinatorial interpretations.

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Definition

Definition: We define $S(n,k)$ by the recursion

$S(n+1,k) = S(n,k-1) + k \cdot S(n,k)$ with initial conditions $S(0,0) = 1$ and $S(n,k) = 0$ if $n < k \leq 0$. We call the number $S(n,k)$ a *Stirling number of the second kind* or a *Stirling set number*.

$S(n,k)$ represents the number of ways of partitioning the set $\{1, 2, \ldots, n\}$ in to k nonempty subsets.

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More Definitions

Definition: We define $s(n,k)$ by the recursion $s(n+1,k) = s(n,k-1) - n \cdot s(n,k)$ with initial conditions $s(0,0) = 1$ and $s(n,k) = 0$ if $n < k \leq 0$. We call the number $s(n,k)$ a *Stirling number of the first kind.*

Definition: We define $c(n,k) = (-1)^{n-k} s(n,k)$, which will satisfy a similar recursion and initial conditions. We call the number $c(n,k)$ a *signless Stirling number of the first kind* or a *Stirling cycle number.*

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A Quick Recap

- $S(n+1,k) = S(n,k-1) + k \cdot S(n,k)$
- $s(n+1,k) = s(n,k-1) - n \cdot s(n,k)$
- $c(n+1,k) = c(n,k-1) + n \cdot c(n,k)$

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Inverse Relations

If we consider the bases for $\mathbb{Q}[x]$, $\{x^m\}_{m\geq 0}$ and $\{(x)\downarrow_m\}_{m\geq 0}$, then, using the recursions, we can show that

$$x^n = \sum_{k=0}^{n} S(n,k)(x)\downarrow_k,$$

and

$$(x)\downarrow_n = \sum_{k=0}^{n} s(n,k)x^k.$$

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Inverse Relations

These two relations lead to the fact that matrices generated by the $S(n,k)$'s and the $s(n,k)$'s are inverses of one another, that is,

$$\sum_{k=0}^{n}\sum_{j=0}^{k} S(n,k)\cdot s(k,j) = \chi(n=j).$$

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A Generating Function

For a fixed $k \in \mathbb{N}$,

$$\sum_{n=k}^{\infty} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

This can be proved using the set partition interpretation of $S(n,k)$, and we will see this identity again later.

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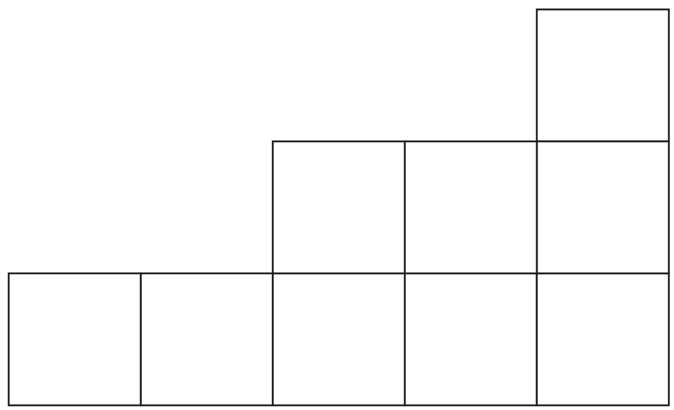
Rook Theory

Define a Ferrers board, $B = F(b_1, b_2, \ldots, b_n)$, to be a board with integer column heights, from left to right, of $b_1 \leq b_2 \leq \cdots \leq b_n$, where $b_1 \geq 0$.



F(1,1,2,2,3)

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Rook Theory

We can define two statistics on these boards, *rook numbers* and *file numbers*.

Definition: We define the k^{th} *rook number* of B, denoted $r_k(B)$, to be the number of ways of placing k rooks on the board B such that no two rooks lie in the same row or column as each other.

Definition: We define the k^{th} *file number* of B, denoted $f_k(B)$, to be the number of ways of placing k rooks on the board B such that no two rooks lie in the same column as each other.

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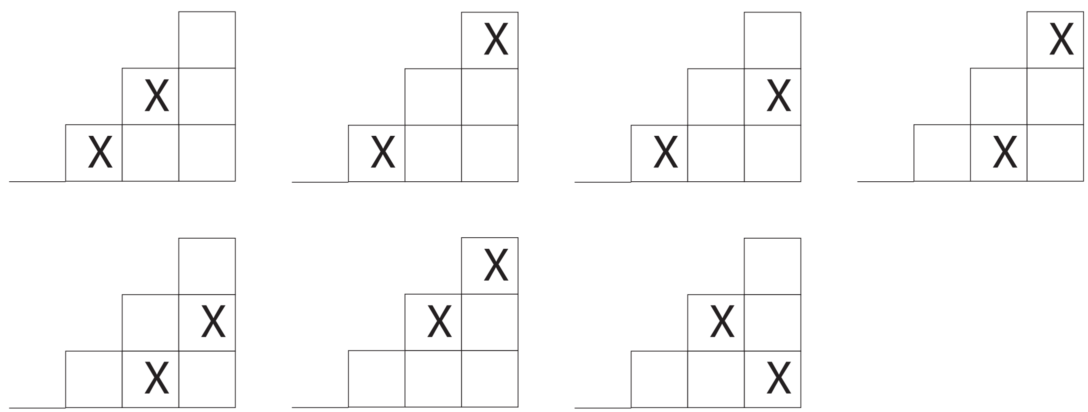
Examples of Rook Numbers

Consider the board $B = F(0,1,2,3)$. Then $r_0(B) = 1$, $r_1(B) = 6$, $r_2(B) = 7$, and $r_3(B) = 1$.



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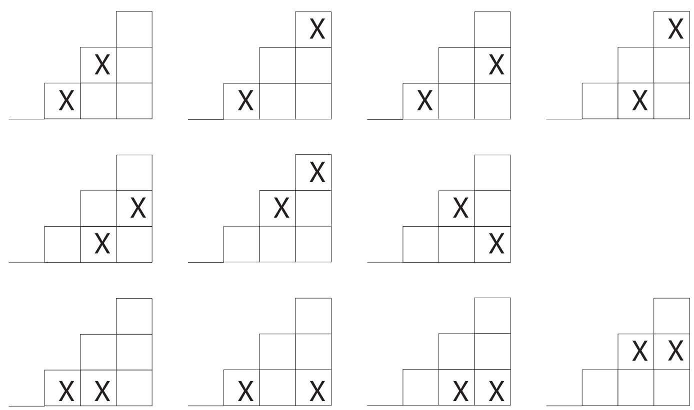
Examples of File Numbers

Consider the board $B = F(0, 1, 2, 3)$. Then $f_0(B) = 1$, $f_1(B) = 6$, $f_2(B) = 11$, and $f_3(B) = 6$.



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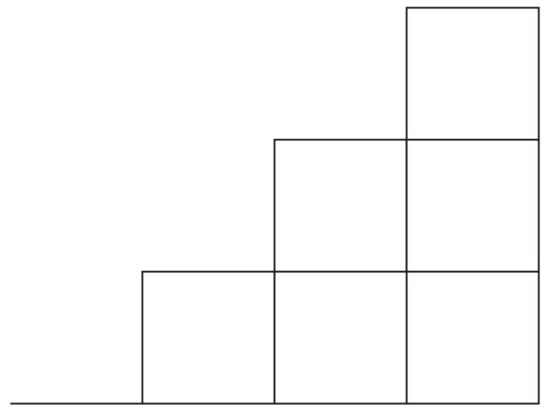
On to the Rook Theory

Definition: We define the n^{th} *staircase board* to be the Ferrers board $B_n = F(0, 1, 2, \ldots, n-1)$. B_4 is shown here.



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On to the Rook Theory

Definition: We define the n^{th} *staircase board* to be the Ferrers board $B_n = F(0, 1, 2, \ldots, n-1)$.

Theorem 1: For all integers n, k, $S(n, k) = r_{n-k}(B_n)$.

Theorem 2: For all integers n, k, $s(n, k) = (-1)^{n-k} f_{n-k}(B_n)$.

Corollary: For all integers n, k, $c(n, k) = f_{n-k}(B_n)$.

Two Proofs of Theorem 1

Theorem 1: For all integers n, k, $S(n, k) = r_{n-k}(B_n)$.

1.) We can show that $S(n, k)$ and $r_{n-k}(B_n)$ satisfy the same recursions with the same initial conditions.

2.) We can make a bijection between the two sets represented by these numbers.

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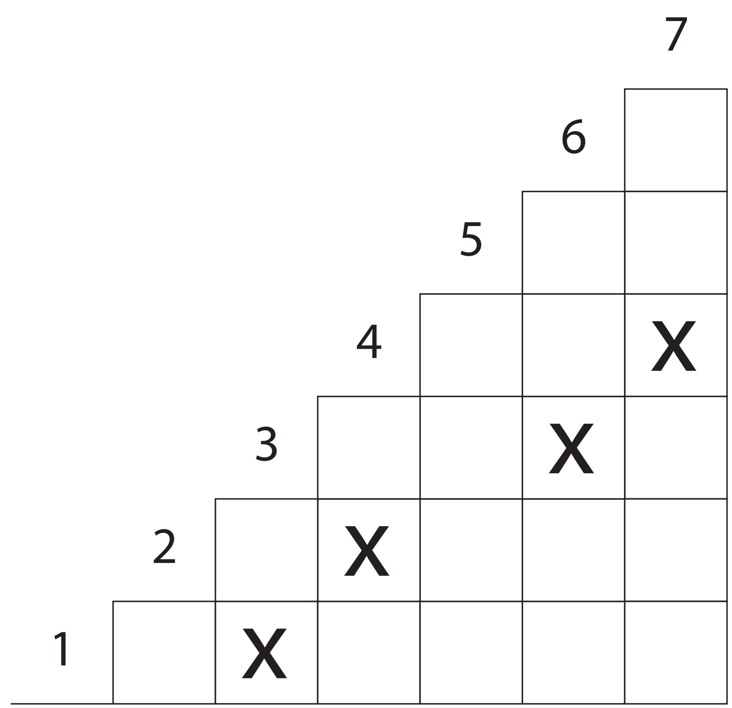
Bijective "Proof" of Theorem 1



$$S(7, 3) = r_4(B_7)$$

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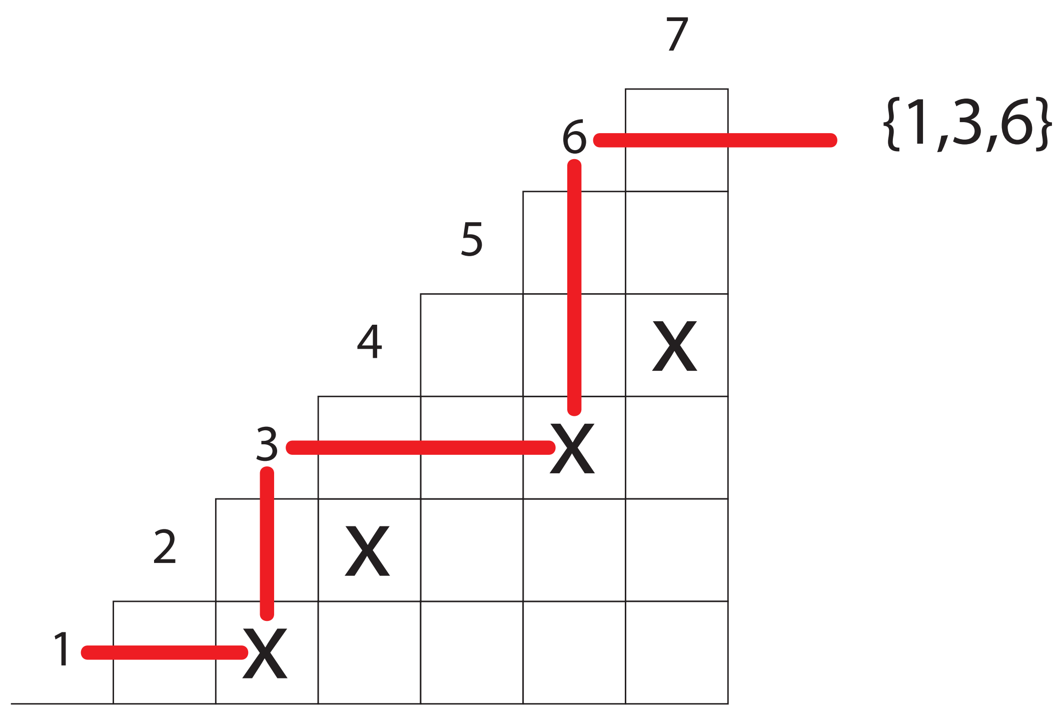
Rook & Stirling Numbers

Bijective "Proof" of Theorem 1



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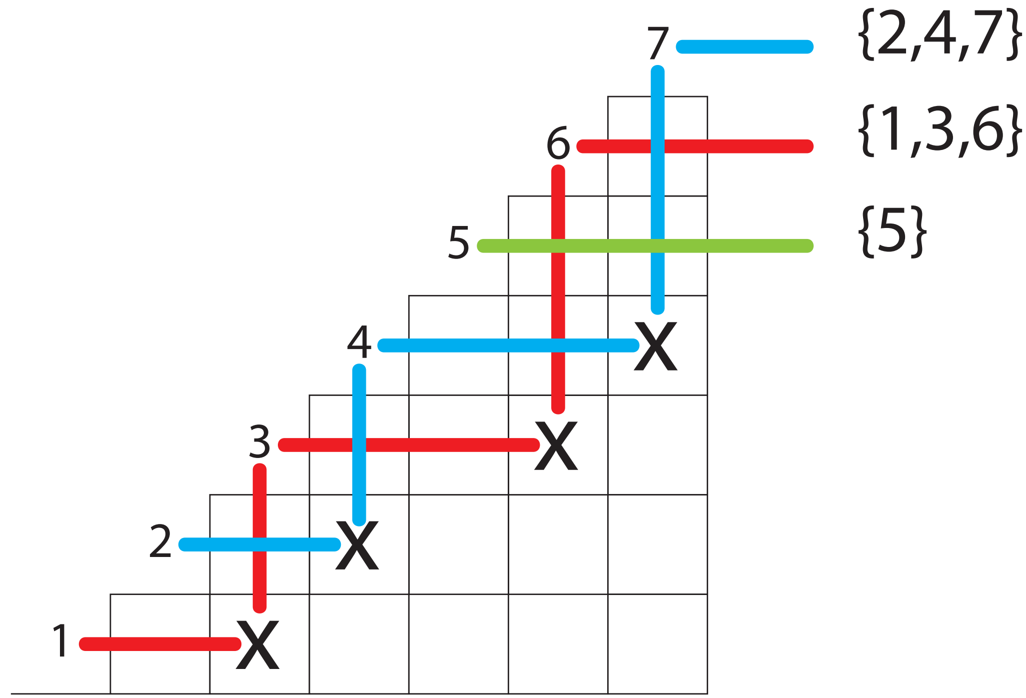

$$S(7,3) = r_4(B_7)$$

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Another Theorem

Theorem 3:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} S(n,k)\cdot s(k,j) = \chi(n=j).$$

In light of our previous theorems, we can rewrite this as follows.

Theorem 3:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} r_{n-k}(B_n)\cdot(-1)^{k-j} f_{k-j}(B_k) = \chi(n=j).$$

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Proof Sketch

1.) Fix n, and then for a given pair of non-negative integers (u, v), consider pairs of boards (B_n, B_u) where B_n contains a rook placement of $n-u$ rooks and B_u contains a file placement of $u-v$ rooks, where this second placement gets a weight of $(-1)^{u-v}$.

2.) Make an appropriate sign-reversing involution.

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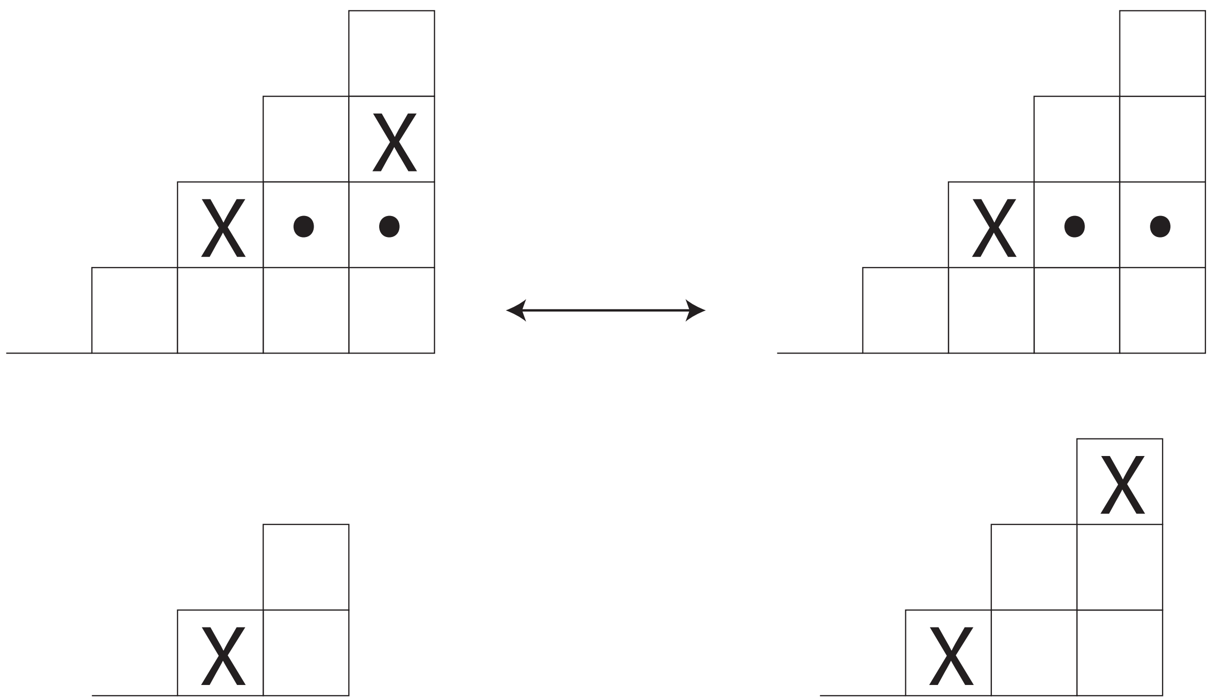
Proof Sketch (continued...)



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The Plan

- We are using the rook theoretic interpretations to give new combinatorial interpretations for our Stirling number identities.
- We want to generalize $S(n,k)$ and $s(n,k)$ in a way that we can find corresponding rook theoretic settings to prove some generalized results.

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Definition

Suppose we are given a polynomial $p(x) \in \mathbb{N}_0[x]$.

Definition: We define $S(n,k,p)$ by the recursion $S(n+1,k,p) = S(n,k-1,p) + p(k)S(n,k,p)$ with initial conditions $S(0,0,p) = 1$ and $S(n,k,p) = 0$ if $n < k \leq 0$. We call the number $S(n,k,p)$ a *poly-Stirling number of the second kind.*

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More Definitions

Definition: We define $s(n,k,p)$ by the recursion $s(n+1,k,p) = s(n,k-1,p) - p(n)s(n,k,p)$ with initial conditions $s(0,0,p) = 1$ and $s(n,k,p) = 0$ if $n < k \leq 0$. We call the number $s(n,k,p)$ a *poly-Stirling number of the first kind*.

Definition: We define $c(n,k,p) = (-1)^{n-k}s(n,k,p)$, which will satisfy a similar recursion and initial conditions. We call the number $c(n,k,p)$ a *signless poly-Stirling number of the first kind*.

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A Quick Recap

- $S(n+1,k,p) = S(n,k-1,p) + p(k)S(n,k,p)$
- $s(n+1,k,p) = s(n,k-1,p) - p(n)s(n,k,p)$
- $c(n+1,k,p) = c(n,k-1,p) + p(n)c(n,k,p)$

Note: These are exactly the standard Stirling numbers when $p(x) = x$. In the case where $p(x) = x^2$ we get the *central factorial numbers**.

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* - See Stanley's *Enumerative Combinatorics, Vol. II* or Riordan's *Combinatorial Identities.*

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x^m-Stirling Numbers

We first consider the case where $p(x) = x^m$ for some $m \in \mathbb{N}$.

- $S(n+1,k,p) = S(n,k-1,p) + k^m S(n,k,p)$
- $s(n+1,k,p) = s(n,k-1,p) - n^m s(n,k,p)$
- $c(n+1,k,p) = c(n,k-1,p) + n^m c(n,k,p)$

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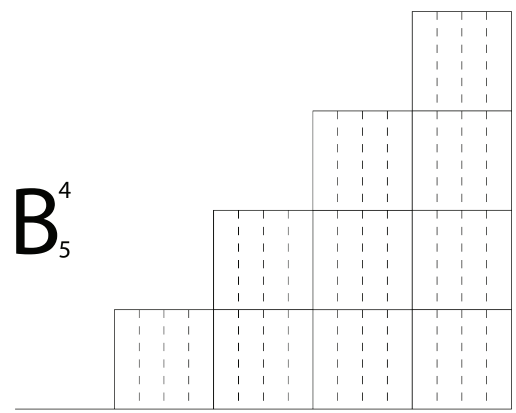
Rook Board Theoretic Interpretations

Given $m, n \in \mathbb{N}$, define B_n^m to be the board B_n with each column partitioned into m subcolumns.



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Rook Theoretic Interpretations

Rook placement rules in the board B_n^m are as follows:

- If a rook is placed in one subcolumn of a column, C, then a rook must be placed in every subcolumn of C, i.e., every column contains either 0 rooks or m rooks.
- Each subcolumn may only contain one rook.
- A rook cancels all cells to its right in its respective subcolumn.

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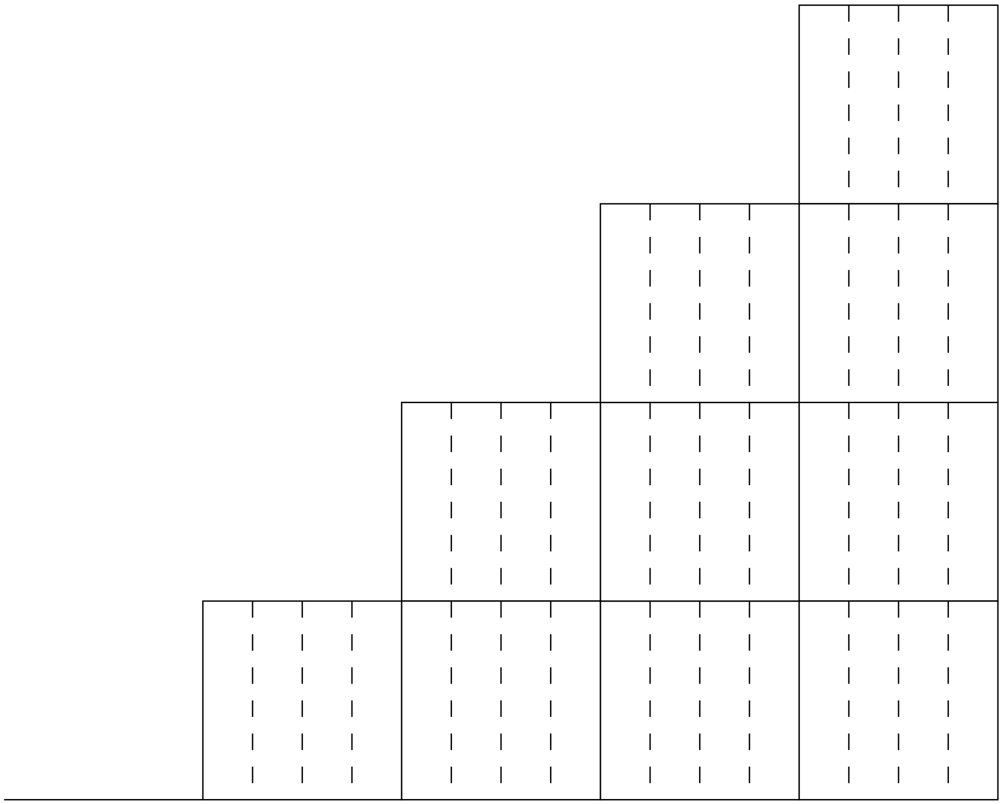
Example of an Attacking Rook Placement in B_5^4



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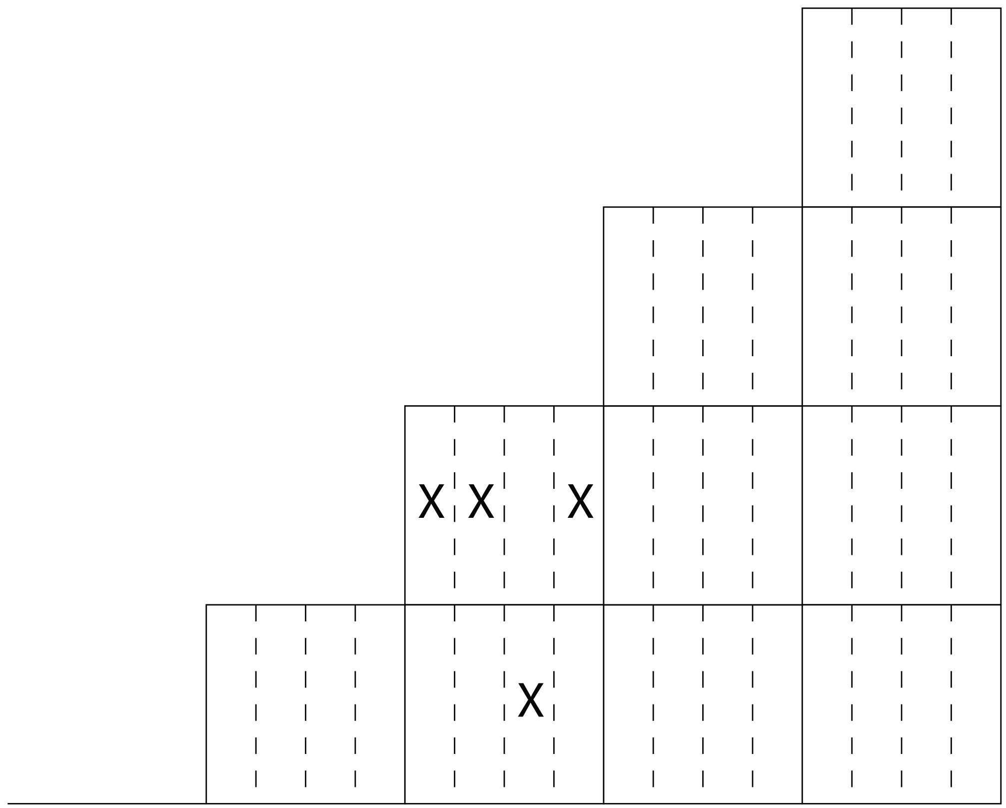

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Rook & Stirling Numbers

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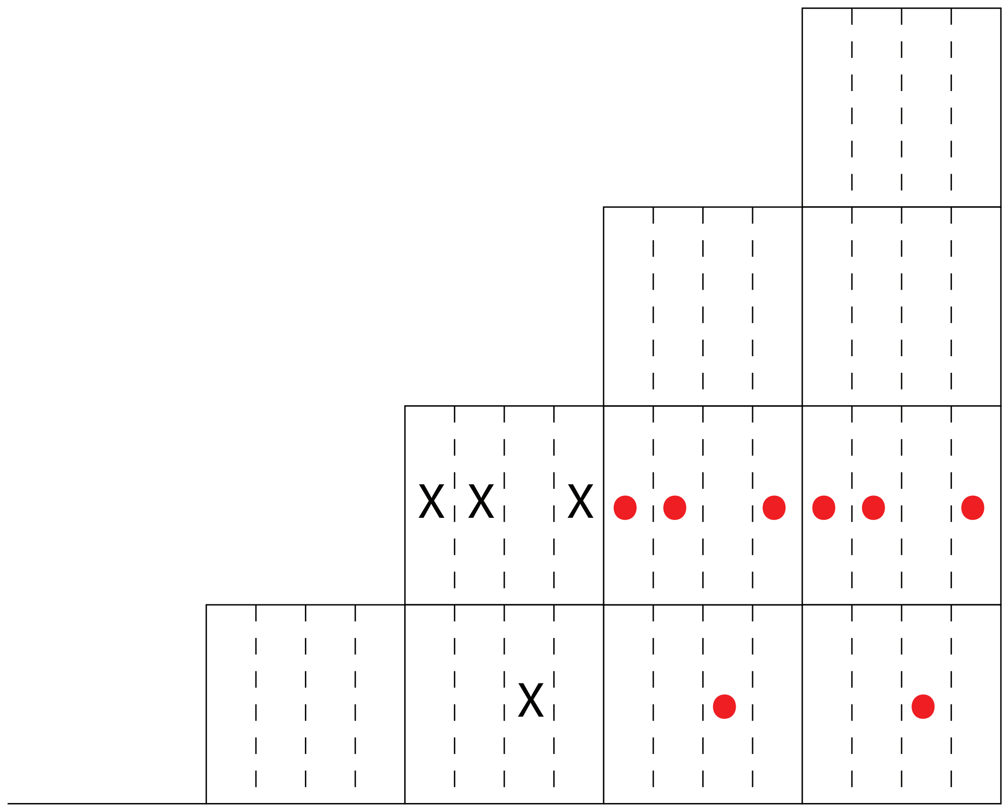

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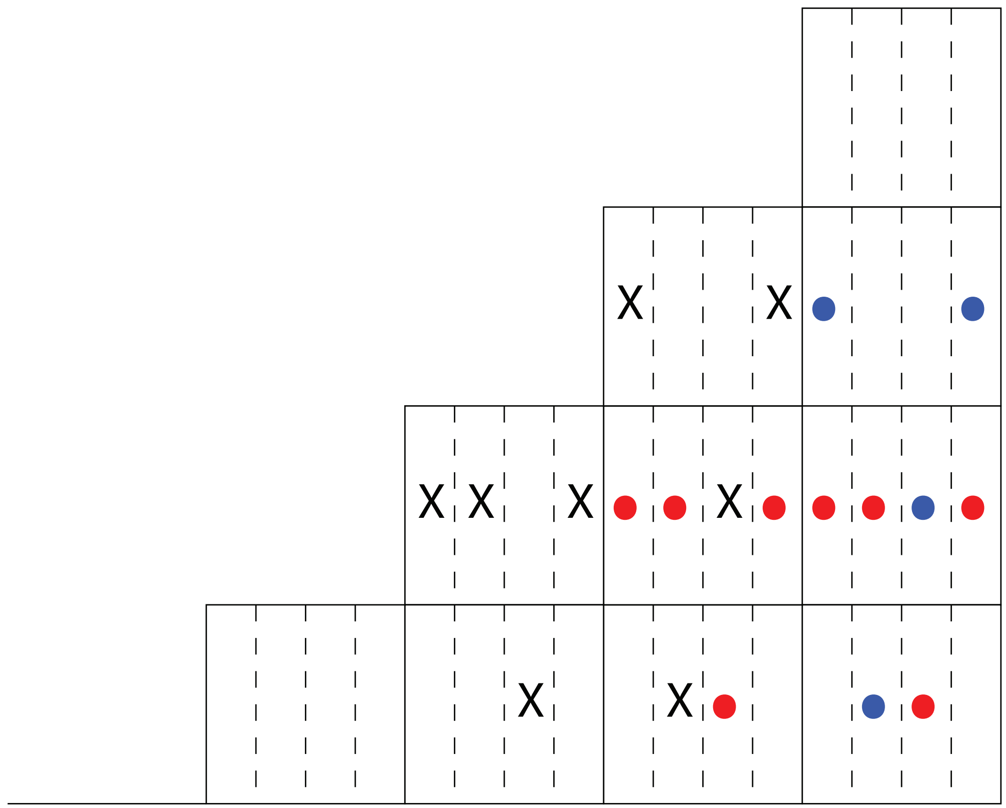
Rook & Stirling Numbers

Example of an Attacking Rook Placement in B_5^4



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Rook Numbers

Definition: The rook number $r_k(B_n^m)$ is the number of ways of placing mk attacking rooks on the board B_n^m.

We notice that $r_{(n+1)-k}(B_{n+1}^m) = r_{n-(k-1)}(B_n^m) + k^m r_{n-k}(B_n^m)$, and we can define $r_0(B_0^m) = 1$. Moreover, $r_{n-k}(B_n^m) = 0$ if $n < k \leq 0$. This is the same recursion as $S(n, k, x^m)$, that is, $S(n, k, x^m) = r_{n-k}(B_n^m)$.

Note: Similar definitions could give rook theoretic interpretations for $c(n, k, x^m)$ and $s(n, k, x^m)$, and all of these can be extended for appropriate $p(x)$.

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Generalizations of Known Formulas

From these rook theoretical interpretations we can get purely combinatorial derivations of the following formulas:

- $(x^m)^n = \sum_{k=0}^{n} S(n,k,x^m)(x^m)(x^m - 1^m)\cdots(x^m - (k-1)^m).$
- $\sum_{k=0}^{n}\sum_{j=0}^{k} S(n,k,x^m)\cdot s(k,j,x^m) = \chi(n=j).$
- $\sum_{n\geq k} S(n,k,x^m)x^n = \dfrac{x^k}{(1-1^m x)(1-2^m x)\cdots(1-k^m x)}.$

Set Partition Interpretations

Like with regular Stirling numbers, we can get set partition interpretations of the numbers $S(n, k, x^m)$.

- $S(n, k, x^m)$ is the number of m-tuples of partitions of $\{1, 2, \ldots, n\}$ in to k parts such that the set of minimal elements of the parts for each partition is the same.
- Example: $S(4, 2, x^2) = 21$

$$\{1\}\{2,3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}, \{1,3,4\}\{2\}$$

$$\{1,2\}\{3,4\}, \{1,2,4\}\{3\}$$

$$\{1,2,3\}\{4\}$$

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Rook & Stirling Numbers

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- Example: $S(4, 2, x^2) = 21 = 4^2 +$

$$\{1\}\{2,3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}, \{1,3,4\}\{2\}$$

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- Example: $S(4, 2, x^2) = 21 = 4^2 + 2^2 +$

$$\{1\}\{2,3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}, \{1,3,4\}\{2\}$$

$$\{1,2\}\{3,4\}, \{1,2,4\}\{3\}$$

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Set Partition Interpretations

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- $S(n, k, x^m)$ is the number of m-tuples of partitions of $\{1, 2, \ldots, n\}$ in to k parts such that the set of minimal elements of the parts for each partition is the same.
- Example: $S(4, 2, x^2) = 21 = 4^2 + 2^2 + 1^2$

$$\{1\}\{2, 3, 4\}, \{1, 3\}\{2, 4\}, \{1, 4\}\{2, 3\}, \{1, 3, 4\}\{2\}$$

$$\{1, 2\}\{3, 4\}, \{1, 2, 4\}\{3\}$$

$$\{1, 2, 3\}\{4\}$$

Rook Setting Corresponding to General $p(x)$

We can define a collection of m-boards and the appropriate notion of rook placements to give rook theoretic interpretations of our most general Poly-Stirling numbers.

We break these into two cases: $p(0) = 0$ and $p(0) \neq 0$. The former works out much nicer!

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Rook & Stirling Numbers

Formulas for General $p(x)$

- $$(p(x))^n = \sum_{k=0}^{n} S(n,k,p) \prod_{j=0}^{k-1} (p(x) - p(j)).$$
- $$\sum_{k=0}^{n} \sum_{j=0}^{k} S(n,k,p) \cdot s(k,j,p) = \chi(n=j).$$
- $$\sum_{n \geq k} S(n,k,p) x^n = \frac{x^k}{(1-p(1)x)(1-p(2)x)\cdots(1-p(k)x)}.$$

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q-Analogues

Let $x \in \mathbb{N}_0$. Then

$$[x]_q = \frac{1-q^x}{1-q} = 1 + q + q^2 + \cdots + q^{x-1}$$

is the *q-analogue of x*.

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Two Types of q-Poly-Stirling Numbers

We can consider two types of q-Poly-Stirling Numbers.

- $S(n+1,k,p)(q) = S(n,k-1,p)(q) + p([k]_q)S(n,k.p)$
- $\overline{S}(n+1,k,p)(q) = \overline{S}(n,k-1,p)(q) + [p(k)]_q\overline{S}(n,k.p)$

We call $S(n,k,p)(q)$ a *Type I q-Poly-Stirling number of the second kind* and $\overline{S}(n,k,p)(q)$ a *Type II q-Poly-Stirling number of the second kind.* Similar analogues can be defined for $c(n,k,p)(q)$, $\overline{c}(n,k,p)(q)$, $s(n,k,p)(q)$, and $\overline{s}(n,k,p)(q)$.

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A Note About Type II

We can reduce the Type II case to a weighted Type I situation related to $S(n,k,x^m)$ via the identity $[x+y]_q = [x]_q + q^x[y]_q$.

Example:

$$\begin{aligned}[x^3+2x+4]_q &= [x^3+2x]_q + q^{x^3+2x}[4]_q \\ &= [x^3]_q + q^{x^3}[2x]_q + q^{x^3+2x}[4]_q\end{aligned}$$

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Some Results

- $(p([x]_q))^n = \sum_{k=0}^{n} S(n,k,p)(q) \prod_{j=0}^{k-1} (p([x]_q) - p([j]_q)).$

- $\sum_{k=0}^{n} \sum_{j=0}^{k} S(n,k,p)(q) \cdot s(k,j,p)(q) = \chi(n=j).$

- $([p(x)]_q)^n = \sum_{k=0}^{n} \overline{S}(n,k,p)(q) \prod_{j=0}^{k-1} ([p(x)]_q - [p(j)]_q).$

- $\sum_{k=0}^{n} \sum_{j=0}^{k} \overline{S}(n,k,p)(q) \cdot \overline{s}(k,j,p)(q) = \chi(n=j).$

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Open Problems

There are many formulas which generalize quite nicely, but also many that do not. For example, we would like to find an exponential generating function for $S(n, p, k)$. This can be done for $p(x) = x^2$, and maybe for $p(x) = x^m$, but it does not seem possible for arbitrary $p(x)$.

Are there (p, q)-analogues of these numbers?

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The End

Thanks for listening, and feel free to ask questions.

(I will post these slides to my web page under "Research" later this week.)

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