Poly-Rook & Stirling Numbers

Brian K. Miceli

Trinity University
Mathematics Department

AMS Sectional
Tulane University
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- The goal here is to present a rook model and then show how this model connects with a generalization of the Stirling numbers.

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The Basics

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We define the n^{th} *staircase board* to be $B_n = F(0, 1, 2, \ldots, n-1)$.

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m-Boards

Given $m \in \mathbb{N}$ and the Ferrers board $B = F(b_1, b_2, \ldots, b_n)$, we define the *m-partition board*, B^m, to be the board B with each column partitioned into m subcolumns.

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We define the k^{th} *m-rook number of B^m* to be the number ways of placing km nonattacking rooks on the board B^m, denoted by $r_k^{(m)}(B^m)$.

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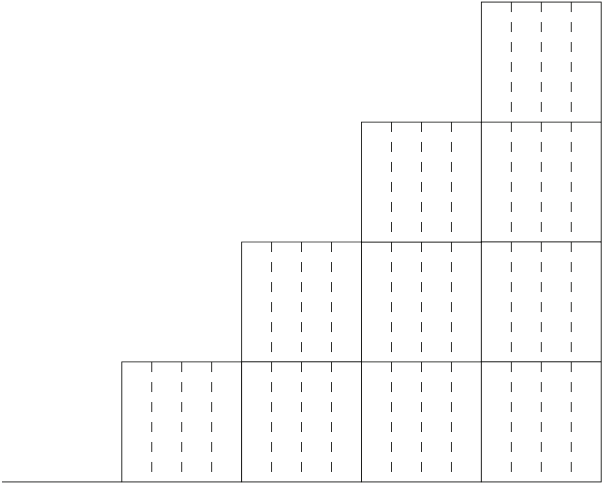
- no rook may be placed in the ℓ^{th} row of the j^{th} subcolumn of the w^{th} column if there is a rook in the ℓ^{th} row of the j^{th} subcolumn of the v^{th} column with $v < w$.

Example

The 5^{th} staircase board with $m = 4$.



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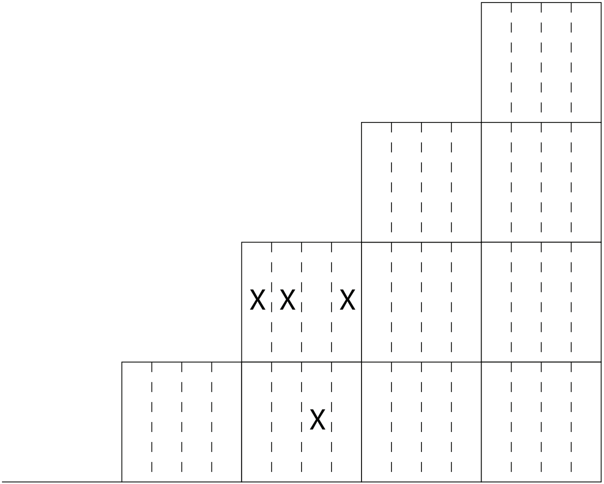

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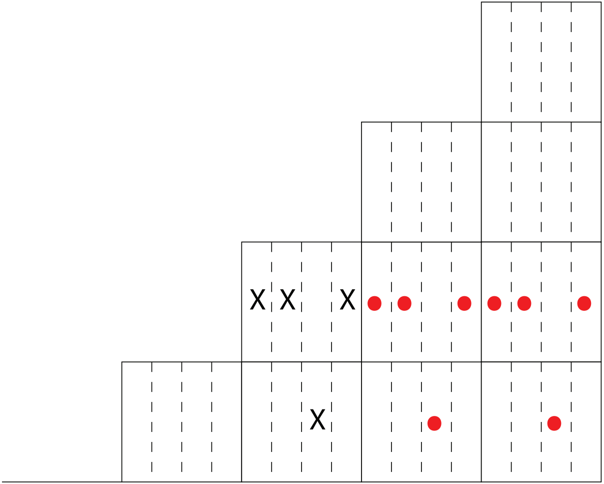

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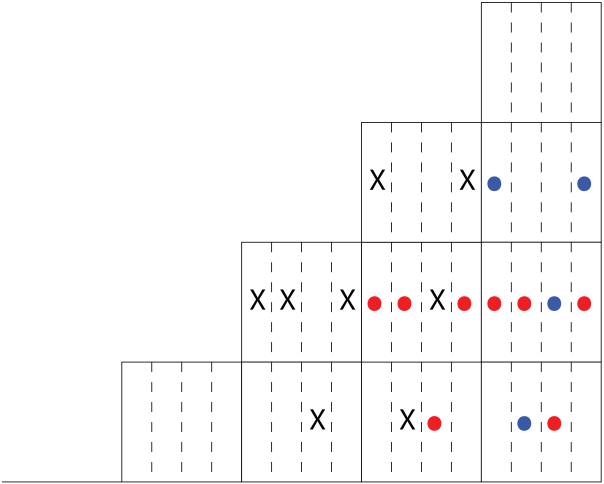

Recursions

Given $B = F(b_1, b_2, \ldots, b_n, b_{n+1})$, define $\overline{B} = F(b_1, b_2, \ldots, b_n)$. Then we have the following recursive relationship for $m \in \mathbb{N}$:

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$$r^{(m)}_{n+1-k}(B^m) = r^{(m)}_{n+1-k}(\overline{B}^m) + [b_{n+1} - (n-k)]^m r^{(m)}_{n-k}(\overline{B}^m).$$

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In the case where we have the staircase board, this relationship becomes:

$$r^{(m)}_{n+1-k}(B^m_{n+1}) = r^{(m)}_{n+1-k}(B^m_n) + k^m r^{(m)}_{n-k}(B^m_n).$$

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Theorem: Suppose $m \in \mathbb{N}$, $x \in \mathbb{R}$, and let $B = F(b_1, b_2, \ldots, b_n)$ be a Ferrers boards.

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(ii) Fix a placement of $n-k$ rooks in B^m and extend. □

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Polyboards

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_t x^t \in \mathbb{N}_0[x]$ with $a_t \neq 0$ and let $B = F(b_1, b_2, \ldots, b_n)$.

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$$B(p(x)) = \{B^{m_{s_1}}, B^{m_{s_2}}, \ldots, B^{m_{s_u}} = B^t\},$$

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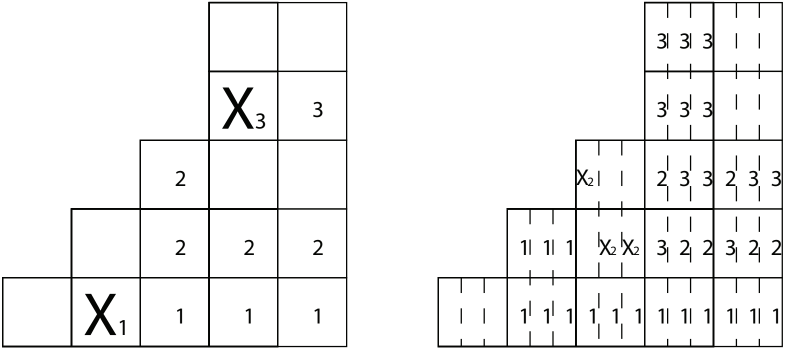
We note that the coefficients of $p(x)$ have nothing to do with the construction of $B(p(x))$.

Example

A placement of rooks in the polyboard $B(p(x))$ with $p(x) = a_1x + a_3x^3$.



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- if a rook in one subcolumn of a column is colored with color c, then every rook in that column is colored with c as well.

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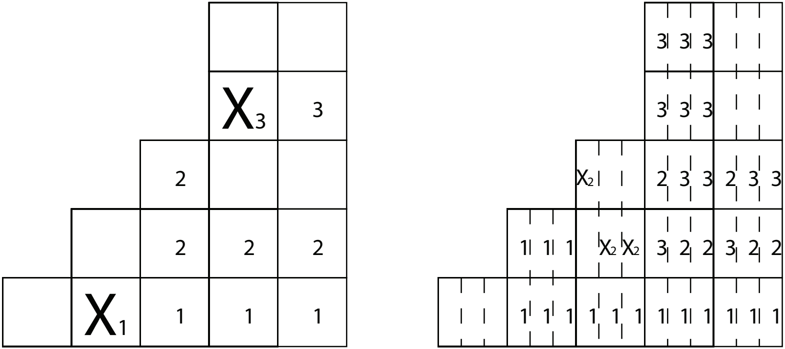

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Definitions

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We call these the *Type I and II poly-Stirling numbers of the Second Kind with respect to* $p(x)$. We see that if $q = 1$ and $p(x) = x$ then we get the classical Stirling numbers of the second kind.

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This gives rook theoretic interpretations to our Stirling numbers, although they still have interpretations in terms of set partitions (in this case, tuples of colored set partitions with minimal element restrictions on the parts).

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$$\frac{t^k}{(1 - tp([1]_q))(1 - tp([2]_q)) \cdots (1 - tp([k]_q))}.$$

Similar formulas hold for $\overline{S}_{n,k}^{p(x)}(q)$.

Future Work

Of particular interest are those poly-Stirling numbers where $p(x) = x^m$ for some $m \in \mathbb{N}$.

1. Allowing for $1 \leq i, j \leq k$, give a rook theoretic proof of the following: $\det(||S^{x^m}_{r+j,i}(1)||) = (k!)^{mr}$.
2. Allowing for $0 \leq i, j \leq k$, give a rook theoretic proof of the following: $\det(||S^{x^m}_{r+i+j,r+i}(1)||) = \prod_{\ell=1}^{k}(r+\ell)^{m\ell}$.

Thank you.