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A Rook Theory Model for Product Formulas & Poly-Stirling Numbers

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The dissertation of Brian K. Miceli is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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This dissertation is dedicated to my beautiful wife-to-be Kristen, who has been there when I needed her the most, and to my wonderful parents, who have always supported everything I do.

TABLE OF CONTENTS

LIST OF FIGURES

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ABSTRACT OF THE DISSERTATION

A Rook Theory Model for Product Formulas & Poly-Stirling Numbers

by

Brian K. Miceli
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Professor Jeffrey B. Remmel, Chair

There are a number of so-called factorization theorems for rook polynomials that have appeared in the literature. For example, Goldman, Joichi, and White [11] showed that for any Ferrers board $B = F(b_1, b_2, \ldots, b_n)$,

$$\prod_{i=1}^{n}(x + b_i - (i-1)) = \sum_{k=0}^{n} r_k(B)(x) \downarrow_{n-k}$$

where $r_k(B)$ is the k^{th} rook number of B and $(x) \downarrow_k = x(x-1)\cdots(x-(k-1))$ is the usual falling factorial polynomial. Similar formulas where $r_k(B)$ is replaced by some appropriate generalization of rook numbers and $(x) \downarrow_k$ is replaced by polynomials like $(x) \uparrow_{k,j} = x(x+j)\cdots(x+j(k-1))$ or $(x) \downarrow_{k,j} = x(x-j)\cdots(x-j(k-1))$ can be found in the work of Goldman and Haglund [10], Remmel and Wachs [24], Haglund and Remmel [14], and Briggs and Remmel [4]. We shall call such formulas generalized product formulas.

In the first part of this dissertation, we develop a new rook theory setting where we can give a uniform combinatorial proof of a generalized product formula which includes all the cases referred to above. That is, given any two sequences of non-negative integers, $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$, and two sign functions $sgn, \overline{sgn} : \{1, \ldots, n\} \to \{-1, 1\}$, we shall define a rook theory

setting and appropriate generalization of rook numbers $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ such that

$$\prod_{i=1}^{n}(x + sgn(i)b_i) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) \prod_{j=1}^{k}(x + (\sum_{s=1}^{j} \overline{sgn}(s)a_s)).$$

We also find q-analogues and (p, q)-analogues of the above formula.

Now, suppose that we are given a polynomial $p(x) \in \mathbb{N}[x]$, and consider the numbers generated by the recursions:

$$\begin{aligned}&s_{0,0}^{p(x)} = 1 \text{ and } s_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and}\\ &s_{n+1,k}^{p(x)} = s_{n,k-1}^{p(x)} - p(n)s_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0,\end{aligned}$$

$$\begin{aligned}&c_{0,0}^{p(x)} = 1 \text{ and } c_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and}\\ &c_{n+1,k}^{p(x)} = c_{n,k-1}^{p(x)} + p(n)c_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0, \text{ and}\end{aligned}$$

$$\begin{aligned}&S_{0,0}^{p(x)} = 1 \text{ and } S_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and}\\ &S_{n+1,k}^{p(x)} = S_{n,k-1}^{p(x)} + p(k)S_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.\end{aligned}$$

In the second part of this dissertation, we discuss various combinatorial interpretations of these numbers, which we call *poly-Stirling numbers*. We give rook theory interpretations for them, find product formulas which involve these poly-Stirling numbers, and we define multiple q-analogues of these numbers.

Chapter 1

Classical Rook Theory, Stirling Numbers, $\&$ Q-Analogues

1.1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. For any positive integer a, we will set $[a] := \{1, 2, \ldots, a\}$. We let $\mathcal{B}_n = [n] \times [n]$ be the n by n array of squares (like a chess board). We number the rows $\mathcal{B}_n$ from bottom to top and the columns of $\mathcal{B}_n$ from left to right with the numbers $1, \ldots, n$. We will refer to the square or cell in the i^{th} row and j^{th} column of $\mathcal{B}_n$ as the (i, j)-th cell of $\mathcal{B}_n$. A *rook board* B is any subset of $\mathcal{B}_n$. If $B \subseteq \mathcal{B}_n$ is rook board consisting of the first b_i cells of column i for $i = 1, \ldots, n$, then we will write $B = F(b_1, \ldots, b_n)$ and refer to B as a staircase board. In the special case that $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq n$, we will say that $B = F(b_1, b_2, \ldots, b_n)$ is a *Ferrers board*. For example, $F(1, 2, 2, 4)$ is pictured in Figure 1.1.

Given a board $B \subseteq \mathcal{B}_n$, we let $\mathcal{N}_k(B)$ denote the set of all placements $\mathbb{P}$ of k rooks in B such that no two rooks in $\mathbb{P}$ lie in the same row or column. We

Figure 1.1: A Ferrers board $B = F(1, 2, 2, 4) \subseteq \mathcal{B}_n$, with $n = 4$.

will refer to such a $\mathbb{P}$ as a *placement of k non-attacking rooks* in B. Similarly, we let $\mathcal{F}_k(B)$ denote the set of all placements Q of k rooks in B such that no two rooks in Q lie in the same column. We will refer to such a Q as a *file placement* of k rooks in B. Thus in a file placement Q, we do allow the possibility that two rooks can lie in the same row. We then define the k^{th} *rook number of B*, $r_k(B)$, by setting $r_k(B) := |\mathcal{N}_k(B)|$. Similarly, we define the the k^{th} *file number of B*, $f_k(B)$ by setting $f_k(B) := |\mathcal{F}_k(B)|$. If $B = F(b_1, \ldots, b_n)$, then we shall sometimes write $r_k(b_1, b_2, \ldots, b_n)$ for $r_k(B)$ and $f_k(b_1, b_2, \ldots, b_n)$ for $f_k(B)$.

Given $x \in \mathbb{N}$, define $x\downarrow_n = x(x-1)\cdots(x-(n-1))$ and $x \uparrow_n = x(x+1)\cdots(x+(n-1))$. For each $B \subseteq \mathcal{B}_n$ and each $x \in \mathbb{N}$, we define two polynomials.

$$\begin{aligned} R_{n,B}(x) &= \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k, \text{ and} && (1.1) \\ F_{n,B}(x) &= \sum_{k=0}^{n} f_{n-k}(B)x^k. && (1.2) \end{aligned}$$

We shall refer to $R_n(B)$ as the *n-th rook polynomial of B* and $F_n(B)$ as the *n-th file polynomial of B*.

Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ in the symmetric group S_n, we shall identify σ with the placement $\mathbb{P}_\sigma = \{(1,\sigma_1),(2,\sigma_2),\ldots,(n,\sigma_n)\}$. Then the *hit number*, $h_k(B)$, is the number of $\sigma \in S_n$ such that the placement $\mathbb{P}_\sigma$ intersects the board in exactly k cells.

The rook numbers, the file numbers, and the hit numbers of rook boards have been studied extensively by combinatorialists. For example, there are three basic identities that hold for these numbers.

$$\begin{aligned}\sum_{k=0}^{n} h_k(B)x^k &= \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k, &(1.3)\\ \prod_{i=1}^{n}(x+b_i-(i-1)) &= \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k, \text{ and} &(1.4)\\ \prod_{i=1}^{n}(x+b_i) &= \sum_{k=0}^{n} f_{n-k}(B)x^k. &(1.5)\end{aligned}$$

Identity (1.1) is due to Kaplansky and Riordan [17] and holds for any board $B \subseteq \mathcal{B}_n$. Identity (1.2) holds for all Ferrers boards $B = F(b_1, \ldots, b_n)$ and is due to Goldman, Joichi, and White [11]. Identity (1.3) is due to Garsia and Remmel [7] and holds for all boards of the form $B = F(b_1, \ldots, b_n)$. Formulas (1.2) and (1.3) are examples of what we shall call *product formulas* in rook theory. That is, they say that for a Ferrers board $B = F(b_1, \ldots, b_n)$, the polynomials $R_{n,B}(x)$ and $F_{n,B}(x)$ factor as a product of linear factors.

We note that in the special case where $B := \mathbf{B}_n = F(0, 1, 2, \ldots, n-1)$, Equations (1.2) and (1.3) become

$$\begin{aligned}x^n &= \sum_{k=0}^{n} r_{n-k}(\mathbf{B}_n)(x)\downarrow_k \text{ and} &(1.6)\\ (x)\uparrow_n &= \sum_{k=0}^{n} f_{n-k}(\mathbf{B}_n)x^k. &(1.7)\end{aligned}$$

This shows that $r_{n-k}(\mathbf{B}_n) = S_{n,k}$, where $S_{n,k}$ is the Stirling number of the second kind, and $(-1)^{n-k} f_{n-k}(\mathbf{B}_n) = s_{n,k}$, where $s_{n,k}$ is the Stirling number of the first kind, and thus, we obtain rook theory interpretations for the Stirling numbers of the first and second kind.

There are natural q-analogues of formulas (1.1), (1.2), and (1.3). That is, define $[n]_q = 1 + q + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}$. We then define q-analogues of the factorials and falling factorials by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[x]_q \downarrow_m =$

$[x]_q[x-1]_q\cdots[x-(m-1)]_q$. Garsia and Remmel [7] defined q-analogues of the hit numbers, $h_k(B,q)$, q-analogues of the rook numbers, $r_k(B,q)$, and q-analogues of file numbers, $f_k(B,q)$, for Ferrers boards B so that the following hold:

$$\begin{aligned}\sum_{k=0}^{n} h_k(B,q)x^{n-k} &= \sum_{k=0}^{n} r_{n-k}(B,q)[k]_q!x^k(1-xq^{k+1})\cdots(1-xq^n), && (1.8)\\ \prod_{i=1}^{n}[x+b_i-(i-1)]_q &= \sum_{k=0}^{n} r_{n-k}(B,q)[x]_q\downarrow_k, \quad \text{and} && (1.9)\\ \prod_{i=1}^{n}[x+b_i]_q &= \sum_{k=0}^{n} f_{n-k}(B,q)[x]_q^k. && (1.10)\end{aligned}$$

Finally, we should mention that there are also (p,q)-analogues of such formulas (see the work of Wachs and White [28], Briggs and Remmel [3], and Briggs [2]).

In recent years, a number of researchers have developed new rook theory models which give rise to new classes of product formulas. For example, Haglund and Remmel [14] developed a rook theory model where the analogue of the the rook number $m_k(B)$ counts partial matchings in the complete graph $\mathcal{K}_n$. They defined an analogue of a Ferrers board $\tilde{F}(a_1,\ldots a_{2n-1})$ where $2n-1\geq a_1\geq\cdots\geq a_{2n-1}\geq 0$ and where the nonzero entries in $(a_1,\ldots,a_{2n-1})$ are strictly decreasing and, in their setting, they proved the following identity,

$$\prod_{i=1}^{2n-1}(x+a_{2n-i}-2i+2)=\sum_{k=0}^{2n-1} m_{n-k}(F)x(x-2)(x-4)\cdots(x-2(k-1)). \quad (1.11)$$

Remmel and Wachs [24] defined a more restricted class of rook numbers, $\tilde{r}_k^j(B)$, in their *j-attacking* rook model and proved that for Ferrers boards $B=F(b_1,\ldots,b_n)$, where $b_{i+1}-b_i\geq j-1$ if $b_i\neq 0$,

$$\prod_{i=1}^{n}(x+b_i-j(i-1))=\sum_{k=0}^{n}\tilde{r}_{n-k}^j(B)x(x-j)(x-2j)\cdots(x-(k-1)j). \quad (1.12)$$

Goldman and Haglund [10] developed an *i-creation rook theory model* and proved that for Ferrers boards one has the following identity,

$$\prod_{j=1}^{n}(x+b_i+(j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(i)}(B)x(x+(i-1))\cdots(x+(k-1)(i-1)). \quad (1.13)$$

In all of these new models, the authors proved q-analogues and/or (p,q)-analogues of their product formulas.

The goal of the first chapter is to review the proofs of the basic product formulas described above and their q-analogues. Next we shall look at the special case where $B = F(0,1,\ldots,n-1)$ where the rook numbers and file numbers are equal to the Stirling numbers of the second kind and the signless Stirling numbers of the first kind respectively and develop their q-analogues. Finally, we shall also describe the alternative rook models of Remmel-Wachs [24], Briggs-Remmel[4], Haglund-Remmel [14], and Goldman-Haglund [10].

In the second and third chapters, we shall develop a new rook theory model and prove a quite general product formula and various q-analogues. That is, suppose we are given any two sequences of natural numbers: $\mathcal{B} = \{b_i\}_{i=1}^n, \mathcal{A} = \{a_i\}_{i=1}^n \in \mathbb{N}^n$. Define $A_i = a_1 + a_2 + \cdots + a_i$, the ith partial sum of the a_i's, and let $B = F(b_1, b_2, \ldots, b_n)$ be a rook board. We will also define two functions, sgn and $\overline{sgn}$, such that $sgn, \overline{sgn} : [n] \to \{-1,+1\}$. Our goal is to define a rook theory model with an appropriate notion of the rook numbers $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ such that the following product formula holds:

$$\prod_{i=1}^{n}(x + sgn(i)(b_i)) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) \prod_{j=1}^{k}(x + \sum_{s\leq j} \overline{sgn}(s)(a_s)). \quad (1.14)$$

We will refer to Equation (1.14) as the *general product formula* and the number $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ as the k^{th} *augmented rook number of $\mathcal{B}$ with respect to $\mathcal{A}$, sgn, and $\overline{sgn}$.* This general product formula is new and vastly extends the range of any of the product formulas that have appeared in the literature. In fact, in the $q = 1$ case, we shall see that our general product formula specializes to all the product formulas described above so that our new rook theory model provides a common framework in which we can give a uniform proof of all these product formulas. Finally, we shall also prove q-analogues and p, q-analogues of our general product formula and describe the connection between other q-analogues and (p, q)-analogues of product formulas that have appeared in the literature.

In the final chapter, we shall shall define an extension of Stirling numbers of the first and second which we call the *poly-Stirling numbers*. That is, given any polynomial $p(x)$ with non-negative integer coefficients, we shall define the poly-Stirling numbers of the first kind, $s_{n,k}^{p(x)}$, for all non-negative integers n and and integers k by the following recursions:

$$s_{0,0}^{p(x)} = 1 \text{ and } s_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.15}$$

$$s_{n+1,k}^{p(x)} = s_{n,k-1}^{p(x)} - p(n)s_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \tag{1.16}$$

Similarly, we shall define the poly-Stirling numbers of the second kind, $S_{n,k}^{p(x)}$ for all non-negative integers n and and integers k by the following recursions:

$$S_{0,0}^{p(x)} = 1 \text{ and } S_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.17}$$

$$S_{n+1,k}^{p(x)} = S_{n,k-1}^{p(x)} + p(k)S_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \tag{1.18}$$

Note that in the special case, when $p(x) = x$, the recursions reduce to

$$s_{0,0}^{x} = 1 \text{ and } s_{n,k}^{x} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.19}$$

$$s_{n+1,k}^{x} = s_{n,k-1}^{x} - ns_{n,k}^{x} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0 \tag{1.20}$$

and

Figure 1.2: The board B_x, with $B = F(1, 2, 2, 4)$.

$$S^x_{0,0} = 1 \text{ and } S^x_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n\text{, and} \tag{1.21}$$

$$S^x_{n+1,k} = S^x_{n,k-1} + kS^x_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0 \tag{1.22}$$

which are the usual defining relations for the Stirling numbers of the first and second kind respectively. We shall give a new rook theory model to give a combinatorial interpretation for the poly-Stirling numbers of the first and second kind and show that there are various ways to give a q-analogue of our rook theory model which leads to several distinct q-analogues of the the poly-Stirling numbers of the first and second kind.

1.2 Two Classical Product Formulas

In this section, we shall give the proofs of the product formulas of Goldman, Joichi, and White [11] and Garsia and Remmel [7] described in the introduction.

Suppose that we are given a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$. We then define the board B_x to be the board B with x rows appended below, and each row containing n columns, as illustrated in Figure 1.2. We will call the part of the board which we attached below B the *x-part* of B_x, and we will call the part of the board that separates the x-part from B the *bar*, as in Figure 1.2. Then we let $\mathcal{N}_k(B_x)$ denote the set of all placements $\mathbb{P}$ of k rooks in B_x such that no two rooks in $\mathbb{P}$ lie in the same row or column and we let $\mathcal{F}_k(B_x)$ denote the set of all placements Q of k rooks in B_x such that no two rooks in Q lie in the same column.

Theorem 1.1. *(Goldman-Joichi-White [11]). Suppose $x, n \in \mathbb{N}$ with $x \geq n$. If $B = F(b_1, \ldots, b_n)$ is any Ferrers board, then*

$$\prod_{i=1}^{n}(x+b_i-(i-1)) = \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k . \tag{1.23}$$

Proof: We shall show that (1.23) represents two ways to count $|\mathcal{N}_n(B_x)|$. That is, consider the number of ways that we can place the rooks in each column, starting with the leftmost column and working to the right. In the first column, there will be $x+b_1$ cells in which to place the rook. Then, in the second column, we originally have $x+b_2$ cells to place the rook, but one of those cells has been cancelled since there is a rook to its left and we only allow one rook per row. Thus, in the second column there are $x+b_2-1$ cells to place a rook. Continuing on in this fashion, we can see that in the j^{th} column there will be $x+b_j-(j-1)$ rows in which to place the rook in column j. If we look at all possible combinations over all n columns, we obtain the left-hand side of Equation (1.23).

Next, suppose that we first fix a placement $\mathbb{P}$ of $n-k$ non-attacking rooks above the bar in B_x. We claim that there are $(x)\downarrow_k$ ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{N}_n(B_x)$ such that $Q \cap B = \mathbb{P}$. That is, we want to count the number of ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{N}_n(B_x)$ by placing an additional k rooks below the bar. If we look at the leftmost available column in which to place a rook below the bar, then there will be x possible cells in which to place it. As we move to the right, the next available column in which to place a rook below the bar will have $x-1$ cells left to place the rook below the bar in the first empty column. Continuing on in this way, we see that number of such Q is $x(x-1)\cdots(x-(k-1)) = (x)\downarrow_k$. Thus we see that

$$\begin{aligned}
|\mathcal{N}_n(B_x)| &= \sum_{k=0}^{n} \sum_{\mathbb{P}\in\mathcal{N}_{n-k}(B)} (x)\downarrow_k \\
&= \sum_{k=0}^{n}(x)\downarrow_k \sum_{\mathbb{P}\in\mathcal{N}_{n-k}(B)} 1 \\
&= \sum_{k=0}^{n}(x)\downarrow_k r_{n-k}(B)
\end{aligned}$$

as desired. □

In the same way that we proved Theorem 1.1, we can also prove the following theorem due to Garsia and Remmel [7].

Theorem 1.2. *Suppose* $x \in \mathbb{N}$ *and* $B = F(b_1, b_2, \ldots, b_n)$ *is a staircase board. Then*

$$\prod_{i=1}^{n}(x + b_i) = \sum_{k=0}^{n} f_{n-k}(B)x^k. \tag{1.24}$$

Proof: We shall show that (1.24) represents two ways to count $|\mathcal{F}_n(B_x)|$. That is, consider the number of ways that we can place the rooks in each column, starting with the leftmost column and working to the right. In the first column, there will be $x + b_1$ cells in which to place the rook. It is easy to see that since we are allowed to place two rooks in the same row for file placements, then we have exactly $x + b_i$ ways to place a rook in the i-th column for $i = 1, \ldots, n$. Thus

$$|\mathcal{F}_n(B_x)| = \prod_{i=1}^{n}(x + b_i).$$

Next, suppose that we first fix a file placement $\mathbb{P}$ with $n - k$ rooks above the bar in B_x. We claim that there are x^k ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_n(B_x)$ such that $Q \cap B = \mathbb{P}$. That is, we want to count the number of ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_n(B_x)$ by placing an additional k rooks below the bar. Again, we see that for each empty column, there are exactly x ways to place the rook below in the bar in that column. Thus we see that

$$\begin{aligned}
|\mathcal{F}_n(B_x)| &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{F}_{n-k}(B)} x^k \\
&= \sum_{k=0}^{n} x^k \sum_{\mathbb{P} \in \mathcal{F}_{n-k}(B)} 1 \\
&= \sum_{k=0}^{n} x^k f_{n-k}(B)
\end{aligned}$$

as desired. □

1.3 Stirling Numbers of the First & Second Kind

In this section, we shall define the Stirling numbers of the first and second kind and develop their connections with the file numbers and rook numbers for the board $\mathbf{B}_n = F(0, 1, 2, \ldots, n-1)$.

The *Stirling numbers of the first kind*, $s_{n,k}$, are defined by the following recursions:

$$s_{0,0} = 1 \text{ and } s_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.25}$$

$$s_{n+1,k} = s_{n,k-1} - n s_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \tag{1.26}$$

One can show by induction that for all $n \geq 1$,

$$(x)\downarrow_n = \sum_{k=1}^{n} s_{n,k} x^k. \tag{1.27}$$

If we let $(x) \uparrow_n = x(x+1) \cdot (x+n-1)$ for $n \geq 1$ and $x \uparrow_0 = 1$, then it is easy to see that if we replace x by $-x$ in (1.27) and then multiply both sides by $(-1)^n$, we will obtain

$$(x) \uparrow_n = \sum_{k=1}^{n} (-1)^{n-k} s_{n,k} x^k. \tag{1.28}$$

The numbers $c_{n,k} = (-1)^{n-k} s_{n,k}$ are called the *signless Stirling numbers of the first kind* and they have a simple combinatorial interpretations. Namely, for $n \geq 1$ and $1 \leq k \leq n$, $c_{n,k}$ is the number of permutations σ of the symmetric group S_n such that σ has k cycles. For example, by listing all of the permutations of $[4]$ with 2 cycles, we see that $c_{4,2} = 11$:

$$(1,2)(3,4), (1,3)(2,4), (1,4)(2,3),$$

$$(1,2,3)(4), (1,3,2)(4), (1,2,4)(3), (1,4,2)(3),$$

$$(1,3,4)(2), (1,4,3)(2), (1)(2,3,4), (1)(2,4,3).$$

Note that we have that

$$(x)\uparrow_n = \sum_{k=1}^{n} c_{n,k} x^k \tag{1.29}$$

and the $c_{n,k}$'s are defined by the recursions

$$c_{0,0} = 1 \text{ and } c_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.30}$$

$$c_{n+1,k} = c_{n,k-1} + n s_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \tag{1.31}$$

The *Stirling numbers of the second kind*, $S_{n,k}$, are defined by the following recursions:

$$S_{0,0} = 1 \text{ and } S_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.32}$$

$$S_{n+1,k} = S_{n,k-1} + k S_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0 \tag{1.33}$$

One can again show by induction that for all $n \geq 1$,

$$x^n = \sum_{k=1}^{n} S_{n,k} (x)\downarrow_k. \tag{1.34}$$

The Stirling numbers of the second kind $S_{n,k}$ also have nice combinatorial interpretations in terms of set partitions. That is, for $n \geq 1$ and $1 \leq k \leq n$, $S_{n,k}$ is the number of unordered set partitions of $\{1, \ldots, n\}$ into k parts. As an example, we can see that $S_{4,2} = 7$ by listing all of the partitions of $[4]$ into 2 parts:

$$\{1,2,3\}\{4\}, \{1,2,4\}\{4\}, \{1,3,4\}\{2\}, \{1\}\{2,3,4\},$$

$$\{1,2\}\{3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}.$$

Next let us consider the board $\mathbf{B}_n = F(0,1,2,\ldots,n-1) \subseteq \mathcal{B}_n$ which we call the *staircase board with n columns*. Note that in this case $r_n(\mathbf{B}_n) = f_n(\mathbf{B}_n) = 0$ since there are neither non-attaching rook placements nor file placements in $\mathbf{B}_n$

which have n rooks. Thus if we consider the special cases of Theorem 1.1 and Theorem 1.2 where $B = \mathbf{B}_n$, we get the following identities:

$$\begin{aligned} x^n &= \sum_{k=1}^{n} r_{n-k}(\mathbf{B}_n)(x)\downarrow_k, \text{ and} && (1.35) \\ (x)\uparrow_n &= \sum_{k=1}^{n} f_{n-k}(\mathbf{B}_n)x^k. && (1.36) \end{aligned}$$

It then follows that for $n \geq 1$ and $1 \leq k \leq n$,

$$\begin{aligned} c_{n,k} &= f_{n-k}(\mathbf{B}_n) \text{ and} && (1.37) \\ S_{n,k} &= r_{n-k}(\mathbf{B}_n). && (1.38) \end{aligned}$$

1.4 Q-Rook Theory

1.4.1 Introduction

We define the q-analogues of n, $n!$, $n\downarrow_k$, $n\uparrow_k$, and $\binom{n}{k}$ respectively by

$$\begin{aligned} [n]_q &= 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}, \\ [n]_q! &= [n]_q[n-1]_q \cdots [2]_q[1]_q, \\ [n]_q\downarrow_k &= [n]_q[n-1]_q \cdots [n-(k-1)]_q, \\ [n]_q\uparrow_k &= [n]_q[n+1]_q \cdots [n+(k-1)]_q, \quad \text{and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}. \end{aligned}$$

A common theme in combinatorics is to state and interpret combinatorial formulas which are q-analogues of set theoretic formulas. The idea is to replace ordinary counting by what we call "q-counting". That is, for the objects in the set theoretic formula, we "count" the object by assigning a q-weight. We would like a q-analogue to satisfy the property that by setting $q = 1$, the original formula is obtained. For example, if S_n is the set of permutations of $\{1, \ldots, n\}$ and $R_{1^k,0^{n-k}}$ is the number of rearrangements of k $1'$s and $n-k$ $0'$s, then

$$\begin{aligned} |S_n| &= n! \text{ and} \qquad (1.39) \\ |R_{1^k,0^{n-k}}| &= \binom{n}{k}. \qquad (1.40) \end{aligned}$$

For any sequence $s = s_1 \ldots s_n$ of natural numbers, we let

$$inv(s) = \sum_{1 \leq i < j \leq n} \chi(s_i > s_j)$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Then it is well known that

$$\begin{aligned} \sum_{\sigma \in S_n} q^{inv(\sigma)} &= [n]_q! \text{ and} \qquad (1.41) \\ \sum_{r \in R_{1^k,0^{n-k}}} q^{inv(r)} &= \begin{bmatrix} n \\ k \end{bmatrix}_q. \qquad (1.42) \end{aligned}$$

Thus, q-counting permutations by $q^{inv(\sigma)}$ gives the q-analogue of $n!$ and q-counting rearrangements r of k $1's$ and $(n-k)$ $0's$ via the statistic $q^{inv(r)}$ gives the q-analogue of $\binom{n}{k}$.

In the next section, we shall define appropriate q-weights on placements of non-attacking rooks and file placements to give q-analogues of the rook and file numbers for Ferrers boards and q-analogues of the product formulas.

1.4.2 Q-Rook & File Numbers

In [7], Garsia and Remmel defined a q-analogue formula for rook numbers by *q-counting* rook configurations. In particular, they defined the k^{th} *q-rook number of* B for a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$ by

Figure 1.3: An example of the q-cancellation rules, where $B = F(1, 2, 2, 3, 3, 4, 5)$, $u_B(\mathbb{P}) = 7$, and $\tilde{u}_B(\mathbb{P}) = 1$.

$$r_k(B, q) = \sum_{\mathbb{P} \in \mathcal{N}_k(B)} q^{u_B(\mathbb{P})}, \tag{1.43}$$

where each rook in $\mathbb{P}$ cancels all of the cells to its right and below it and we set $u_B(\mathbb{P})$ to be the number of cells in B which do not contain a rook and which have not been cancelled by a rook in $\mathbb{P}$. For example, if we consider the Ferrers board $B = F(1, 2, 2, 3, 3, 4, 5)$ and the placement $\mathbb{P} = \{(2, 2), (5, 3), (7, 4)\} \in \mathcal{N}_3(B)$. Then $u_B(\mathbb{P}) = 7$ as seen in the left-hand side of Figure 1.3, where we indicate the cancelled cells by placing a "•" in those cells.

We can use the same idea to give a q-weight to any placement $\mathbb{P} \in \mathcal{N}_n(B_x)$ by saying that each rook cancels all cells directly below it and all cells directly to its right. Then as before, we let $u_{B_x}(\mathbb{P})$ equal the number of cells in B_x which do not contain a rook and which have not been cancelled by a rook in $\mathbb{P}$. This given, the following q-analogue of Theorem 1.1 was proved by Garsia and Remmel [7].

Theorem 1.3. *Let $x, n \in \mathbb{N}$ with $x \geq n$ and suppose that $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board. Then*

$$\prod_{i=1}^{n} [x + b_i - (i-1)]_q = \sum_{k=0}^{n} r_{n-k}(B, q)[x]_q\downarrow_k. \tag{1.44}$$

Proof: To prove Theorem 1.3, we will compute the sum

$$S(q) = \sum_{\mathbb{P} \in \mathcal{N}_n(B_x)} q^{u_{B_x}(\mathbb{P})} \tag{1.45}$$

in two different ways.

First we will place rooks in the board B_x column by column from left to right. For the leftmost column there are $x + b_1$ cell in which to place the rook and so the contribution of the leftmost column to *S(q)* by placing rooks in the top cell and proceeding downwards is, respectively, 1, q, q^2, …, q^{x+b_1-1}. Thus the total contribution to *S(q)* from the leftmost column is $[x + b_1]_q$. Applying the same argument to the second column, we can see that there are only $x + b_2 - 1$ cells to place a rook, since one cell has been cancelled by the rook placed in the first column, and so the rook placed in the second column will contribute a q-count of $[x + b_2 - 1]_q$ to *S(q)*. Thus, if we were to look at the q-count for the j^{th} column of B_x, we would see that there were originally $x + b_j$ cells to place a rook, but $j - 1$ of those cells have been cancelled by previously placed rooks, so the total contribution of the j^{th} column to S will be $[x + b_j - (j - 1)]_q$. Continuing in this way over all n columns we get that

$$\textit{S(q)} = \prod_{i=1}^{n} [x + b_i - (i - 1)]_q.$$

Next suppose that for some fixed integer k, $0 \leq k \leq n$, we pick a placement $\mathbb{P} \in \mathcal{N}_{n-k}(B)$. We wish to compute

$$\textit{S}(\mathbb{P},\textit{q}) = \sum_{\substack{\mathcal{Q} \in \mathcal{N}_n(B_x) \\ \mathcal{Q} \cap B = \mathbb{P}}} q^{u_{B_x}(\mathcal{Q})}.$$

In order to extend $\mathbb{P}$ to a placement $Q \in \mathcal{N}_n(B_x)$ such that $Q \cap B = \mathbb{P}$, we must place the remaining k non-attacking rooks below the bar. Notice that the q-weight of the cells in the board B is exactly $q^{u_B(\mathbb{P})}$. Further note that in each column containing a rook in the board B, all of the cells in the adjoined x rows will be cancelled. Applying the same argument as above, we find that if $n - k$ rooks are placed above the bar, then the contribution to $S(\mathbb{P}, q)$ from the empty columns below the bar, moving from left to right, will be $[x]_q$, $[x - 1]_q$, …, $[x - (k - 1)]_q$ respectively. That is,

$$S(\mathbb{P}, q) = q^{u_B(\mathbb{P})}[x]_q\!\downarrow_k.$$

Summing over all k and all placements of $n - k$ rooks in B yields

$$\begin{aligned} S(q) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}(B)} S(\mathbb{P}, q) \\ &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}(B)} q^{u_B(\mathbb{P})} [x]_q \downarrow_k \\ &= \sum_{k=0}^{n} r_{n-k}(B, q) [x]_q \downarrow_k . \end{aligned}$$

□

We now define *q-file numbers* by setting

$$f_k(B,q) = \sum_{\mathbb{P} \in \mathcal{F}_k(B)} q^{z_B(\mathbb{P})}, \tag{1.46}$$

where each rook in $\mathbb{P}$ cancels only those cells which lie below it in B. We then set $z_B(\mathbb{P})$ to be the number of cells in B which do not contain a rook and are not cancelled by the rooks in $\mathbb{P}$. In Figure 1.4, we indicate the cancelled cells by placing a "•" in those cells. Thus in Figure 1.4, $B = F(2,2,3,4,4,5)$ and for the placement $\mathbb{P} = \{(1,2),(3,3),(5,2)\} \in \mathcal{F}_3(B)$, we have that $z_B(\mathbb{P}) = 13$. We can extend this statistic to the board B_x by saying that each rook in $\mathbb{P}$ cancels only those cells which lie below it in B_x. We then set $z_{B_x}(\mathbb{P})$ to be the number of cells in B_x which do not contain a rook and are not cancelled by the rooks in $\mathbb{P}$.

Theorem 1.4. *Let $x \in \mathbb{N}$ and suppose that $B = F(b_1, b_2, \ldots, b_n)$ is any rook board. Then*

$$\prod_{i=1}^{n} [x + b_i]_q = \sum_{k=0}^{n} f_{n-k}(B,q)([x]_q)^k. \tag{1.47}$$

Proof: To prove Theorem 1.4, we will compute the sum

$$F(q) = \sum_{\mathbb{P} \in \mathcal{F}_n(B_x)} q^{z_{B_x}(\mathbb{P})} \tag{1.48}$$

in two different ways.

First we will place rooks in the board B_x column by column from left to right. It is easy to see that in the i-th column there are $x + b_i$ cell in which to place the rook and so the contribution of the cell in the i-th column to *F(q)* by placing rooks in the top cell and proceeding downwards are, respectively, 1, q, q^2, $\ldots$, q^{x+b_i-1}. Thus the total contribution to *S(q)* from the i-th column is $[x+b_i]_q$. Thus,

$$\mathit{F(q)} = \prod_{i=1}^{n} [x + b_i]_q. \tag{1.49}$$

Next suppose that for some fixed integer k, $0 \leq k \leq n$, we pick a placement $\mathbb{P} \in \mathcal{F}_{n-k}(B)$. We wish to compute

$$\mathit{F}(\mathbb{P},\mathit{q}) = \sum_{\substack{\mathcal{Q}\in\mathcal{F}_n(B_x)\\ \mathcal{Q}\cap B=\mathbb{P}}} q^{z_{B_x}(\mathcal{Q})}.$$

In order to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_n(B_x)$ such that $Q \cap B = \mathbb{P}$, we must place the remaining k non-attacking rooks below the bar. Notice that the q-weight of the cells in the board B is exactly $q^{z_B(\mathbb{P})}$. Further note that in each column containing a rook in the board B, all of the cells in the adjoined x rows will be cancelled. Applying the same argument as above, we find that if $n - k$ rooks are placed above the bar, then the contribution to $F(\mathbb{P}, q)$ from each empty column below the bar will be a factor of $[x]_q$. That is,

$$F(\mathbb{P}, q) = q^{z_B(\mathbb{P})}[x]_q^k.$$

Summing over all k and all file placements of $n - k$ rooks in B yields

Figure 1.4: An example of the q-cancellation rules, where $B = F(2,2,3,4,4,5)$, $z_B(\mathbb{P}) = 13$, and $\tilde{z}_B(\mathbb{P}) = 2$.

$$
\begin{aligned}
F(q) &= \sum_{k=0}^{n} \sum_{\mathbb{P}\in\mathcal{F}_{n-k}(B)} F(\mathbb{P}, q) \\
&= \sum_{k=0}^{n} \sum_{\mathbb{P}\in\{_{n-k}(B)} q^{u_B(\mathbb{P})}[x]_q^k \\
&= \sum_{k=0}^{n} f_{n-k}(B,q)[x]_q^k.
\end{aligned}
$$

□

An alternate definition of q-rook numbers and q-file numbers are the following ones:

$$\tilde{r}_k(B,q) = \sum_{\mathbb{P}\in\mathcal{N}_k(B)} q^{\tilde{u}_B(\mathbb{P})} \tag{1.50}$$

and

$$\tilde{f}_k(B,q) = \sum_{\mathbb{P}\in\mathcal{F}_k(B)} q^{\tilde{z}_B(\mathbb{P})}. \tag{1.51}$$

Here, for any placement $\mathbb{P}$ of non-attacking rooks in B, the only difference between $\tilde{u}_B(\mathbb{P})$ and $u_B(\mathbb{P})$ is that $\tilde{u}_B(\mathbb{P})$ counts only the number of uncancelled cells which lie above a rook in $\mathbb{P}$. The difference between $\tilde{u}_B(\mathbb{P})$ and $u_B(\mathbb{P})$ is pictured in Figure 1.3, where the diagram on the left shows the placement counted with respect to the statistic u_B and the diagram on the right shows the placement counted with respect to the statistic $\tilde{u}_B$. Similarly, for any file placement Q in B, the only difference between $\tilde{z}_B(Q)$ and $z_B(Q)$ is that for $\tilde{z}_B(Q)$ counts only the number of uncancelled cells which lie above a rook in Q. The difference between $\tilde{z}_B(Q)$ and $z_B(Q)$ is pictured in Figure 1.4, where the diagram on the left shows the placement counted with respect to the statistic z_B and the diagram on the right shows the placement counted with respect to the statistic $\tilde{z}_B$.

The statistics $\tilde{u}_B$ and $\tilde{z}_B$ are not convenient for proving product formulas like the ones in Theorems 1.3 and 1.4, but they are useful for giving different q-analogues the Stirling numbers of the first and second kind which we will describe in the next section.

1.4.3 Q-Stirling Numbers of the Second Kind & a Formula of Frobenius

In this section, we will discuss *q-Stirling numbers of the second kind*. Generally speaking, there are basic two types of q-Stirling numbers of the second kind. The first type q-Stirling numbers of the second kind that we shall discuss were introduced by Gould [12]. These are the polynomials defined by the recursions

$$G_{n+1,k}(q) = G_{n,k-1}(q) + [k]_q G_{n,k}(q), \tag{1.52}$$

with initial conditions $G_{0,0}(q) = 1$ and $G_{n,k}(q) = 0$ for $n < 0$, $k < 0$, or $n < k$.

Theorem 1.5. *For all $n \geq 1$ and $0 \leq k \leq n$, $G_{n,k}(q) = \tilde{r}_{n-k}(\boldsymbol{B}_n, q)$.*

Proof To prove that $G_{n,k}(q) = \tilde{r}_{n-k}(\mathbf{B}_n, q)$ for all $n \geq 1$ and $0 \leq k \leq n$, first observe that $\tilde{r}_{n-0}(\mathbf{B}_n, q) = 0$ for all $n \geq 1$ since there are no placements of n non-attacking rooks in $\mathbf{B}_n$. It is easy to prove by induction that $G_{n,k}(q) = 0$ for all $n \geq 1$.

Next consider the set of placements $\mathcal{N}_{n+1-k}(\mathbf{B}_{n+1})$. They can be partitioned into two sets $Last_{k,n+1}$ consisting of those placements $\mathbb{P} \in \mathcal{N}_{n+1-k}(\mathbf{B}_{n+1})$ which have a rook in the last column and $Nolast_{k,n+1}$ consisting of those placements $\mathbb{P} \in \mathcal{N}_{n+1-k}(\mathbf{B}_{n+1})$ which do not have a rook in the last column. Now a placement $\mathbb{P} \in Nolast_{k,n+1}$ consists of placement of $n-(k-1)$ rooks in B_n plus an empty last column and, hence, it is easy to see that

$$\sum_{\mathbb{P} \in Nolast_{k,n+1}} q^{\tilde{u}_{\mathbf{B}_{n+1}}(\mathbb{P})} = \sum_{\mathbb{P} \in \mathcal{N}_{n-(k-1)}(\mathbf{B}_n)} q^{\tilde{u}_{\mathbf{B}_n}(\mathbb{P})} = \tilde{r}_{n-(k-1)}(\mathbf{B}_n, q).$$

Now a placement $\mathbb{P} \in Last_{k,n+1}$ consists of a placement Q_P of $n-k$ rooks in $\mathbf{B}_n$ plus a rook in the last column. It is easy to see that if we fix a placement $Q \in \mathcal{N}_{n-k}(\mathbf{B}_n)$, there are exactly k ways to extend Q to a placement $\mathbb{P} \in Last_{k,n+1}$ since the last column of $\mathbf{B}_{n+1}$ has height n and each of the $n-k$ rooks in Q cancel the cell in its row in the last column. It follows that we let Q_i denote the result of starting with the placement Q and then placing a rook in i-th available row which does not contain a cancelled cell, reading from top to bottom, then $Q_1, \ldots, Q_k$ represent all ways to extend Q to a placement in $Last_{k,n+1}$. Moreover, since there there will be an extra $i-1$ uncancelled cells above the rook in the last column of Q_i, then

$$\tilde{u}_{\mathbf{B}_{n+1}}(Q_i) = i - 1 + \tilde{u}_{\mathbf{B}_n}(Q).$$

It follows that

$$\sum_{\mathbb{P} \in Last_{k,n+1}} q^{\tilde{u}_{\mathbf{B}_{n+1}}(\mathbb{P})} = \sum_{Q \in \mathcal{N}_{n-(k-1)}(\mathbf{B}_n)} (1 + q + \cdots + q^{k-1}) q^{\tilde{u}_{\mathbf{B}_n}(Q)} = [k]_q \tilde{r}_{n-k}(\mathbf{B}_n, q).$$

Thus,

$$\tilde{r}_{n+1-k}(\mathbf{B}_{n+1}, q) = \tilde{r}_{n-(k-1)}(\mathbf{B}_n, q) + [k]_q \tilde{r}_{n-k}(\mathbf{B}_n, q). \tag{1.53}$$

We can know use this recursion to show by induction that $G_{n,k}(q) = \tilde{r}_{n-k}(\mathbf{B}_n, q)$ for all $n \geq 1$ and $1 \leq k \leq n$. □

The second type of q-Stirling numbers are the polynomials denoted by $S_{n,k}(q)$, which are defined by the recursion

$$S_{n+1,k}(q) = q^{k-1} S_{n,k-1}(q) + [k]_q S_{n,k}(q), \tag{1.54}$$

with identical initial conditions as the $G_{n,k}(q)$. One can see that $S_{n,k}(q)$ and $G_{n,k}(q)$ must be related by

$$S_{n,k}(q) = q^{\binom{k}{2}} G_{n,k}(q).$$

The $S_{n,k}(q)$ have been given various combinatorial interpretations in terms of partitions, or equivalently, in terms of restricted growth functions(see [21], [30], and [29]), and $0,1$-tableaux (see [19] and [20]). In [7], Garsia and Remmel gave a combinatorial interpretation for $S_{n,k}(q)$ by q-counting the configurations of $n-k$ non-attacking rooks in the staircase board $\mathbf{B}_n = F(0,1,2,\ldots,n-1)$. That is, they proved the following result by using essentially the same argument that was used to prove Theorem 1.5.

Theorem 1.6. *For all* $n \geq 1$ *and* $0 \leq k \leq n$,

$$S_{n,k}(q) = r_{n-k}(\boldsymbol{B}_n, q). \tag{1.55}$$

Note that in the special case of Theorem 1.3 where $B = \mathbf{B}_n$, we have that

$$([x]_q)^n = \sum_{k=0} S_{n,k}(q)[x]_{q}\downarrow_{k}. \tag{1.56}$$

Finally, we note that one can also prove a q-analogue of the Frobenius formulas using the $S_{n,k}(q)$'s. That is, define B_∞ to be a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$ with infinitely many rows of length n appended below the bar. Given a placement $\mathbb{P} \in \mathcal{N}_n(B_\infty)$, we label the rows below the bar of B_∞ with the numbers $1, 2, \ldots$, and we define $max_{B_\infty}(\mathbb{P})$ to be the row below the bar containing the bottommost rook in $\mathbb{P}$, with the condition that if $\mathbb{P}$ contains no rooks below the bar, then $max_{B_\infty}(\mathbb{P}) = 0$. For $B = \mathbf{B}_n$, Garsia and Remmel [7] gave a combinatorial proof of the following q-analogue of a formula of Frobenius [6]:

$$\frac{1}{1-x}\sum_{\mathbb{P}\in\mathcal{N}_n(B_\infty)} x^{\max_{B_\infty}(\mathbb{P})} q^{u_{B_\infty}(\mathbb{P})} = \sum_{k=0}^{n} \frac{S_{n,k}\,[k]_q!\,x^k}{(1-x)(1-xq)\cdots(1-xq^k)}$$

$$= \sum_{k\geq 0} x^k\,[k]_q^n \qquad (1.57)$$

$$= \frac{\sum_{\sigma\in S_n} x^{des(\sigma)+1} q^{maj(\sigma)}}{\prod_{i=0}^{n}(1-xq^i)}, \qquad (1.58)$$

where for any permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$,

$$Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}, \qquad (1.59)$$

$$des(\sigma) = |Des(\sigma)|, \text{ and} \qquad (1.60)$$

$$maj(\sigma) = \sum_{i\in Des(\sigma)} i. \qquad (1.61)$$

1.4.4 Q-Stirling Numbers of the First Kind

Similar to the q-Stirling numbers of the second kind, the q-Stirling numbers of the first kind can be defined in two different ways. To start, we will define the *signless q-Stirling numbers of the first kind*. These are the polynomials $Z_{n,k}(q)$ defined by the recursion

$$Z_{n+1,k}(q) = Z_{n,k-1}(q) + [n]_q Z_{n,k}(q), \qquad (1.62)$$

with initial conditions $Z_{0,0}(q) = 1$ and $Z_{n,k}(q) = 0$ for $n < 0$, $k < 0$, or $n < k$. We can then define the *q-Stirling numbers of the first kind*, $z_{n,k}(q)$, by setting

$$z_{n,k}(q) = (-1)^{n-k} Z_{n,k}(q).$$

From (1.62) we find that these polynomials must satisfy

$$z_{n+1,k}(q) = z_{n,k-1}(q) - [n]_q z_{n,k}(q),$$

with the same initial conditions as the $Z_{n,k}(q)$. Then by essentially the same argument used to prove Theorem 1.5, one can prove the following.

Theorem 1.7. *For all* $n \geq 1$ *and* $0 \leq k \leq n$,

$$Z_{n,k}(q) = \tilde{f}_{n-k}(\boldsymbol{B}_n, q). \tag{1.63}$$

Now, an alternate definition for the signless q-Stirling numbers of the first kind, which corresponds with those defined in the previous section, are the polynomials $c_{n,k}(q)$ defined by the recursion

$$c_{n+1,k}(q) = q^n c_{n,k-1}(q) + [n]_q c_{n,k}(q), \tag{1.64}$$

with initial conditions $c_{0,0}(q) = 1$ and $c_{n,k}(q) = 0$ for $k < 0$ and $k > n$. Again, we obtain the q-Stirling numbers of the first kind, $s_{n,k}(q)$, by setting $s_{n,k}(q) = (-1)^{n-k} c_{n,k}(q)$. From (1.64) we find that these polynomials must satisfy the recursion

$$s_{n+1,k}(q) = q^n s_{n,k-1}(q) - [n]_q s_{n,k}(q) \tag{1.65}$$

with the same initial conditions as the $c_{n,k}(q)$. Again, by essentially the same argument used to prove Theorem 1.5, one can prove the following theorem.

Theorem 1.8. *For all* $n \geq 1$ *and* $0 \leq k \leq n$,

$$c_{n,k}(q) = f_{n-k}(\boldsymbol{B}_n, q). \tag{1.66}$$

The special case of Theorem 1.4 when $B = F(0, 1, \ldots, n-1)$ gives

$$[x]_q \uparrow_n = \sum_{k=0}^{n} c_{n,k}(q)([x]_q)^k. \tag{1.67}$$

Similarly, replacing $[x]_q$ by $-[x]_q$ in Equation (1.67) and multiplying both sides by $(-1)^n$, we obtain the following product formula for q-Stirling numbers of the first kind:

$$[x]_q \downarrow_n = \sum_{k=0}^{n} s_{n,k}(q)([x]_q)^k. \tag{1.68}$$

It then follows from (1.56) and (1.68) that we have the following result.

Theorem 1.9. *The lower triangular matrices defined by* $||S_{n,k}(q)||$ *and* $||s_{n,k}(q)||$ *are inverses of one another.*

1.5 (P, Q)-Analogues

Define the (p, q)-analogue of $n \in \mathbb{N}$ to be $[n]_{p,q}$ by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + qp^{n-2} + \cdots q^{n-1}p + q^{n-1}. \tag{1.69}$$

One can then use $[n]_{p,q}$ to define the (p, q)-analogues of $n!$, $(n)\downarrow_k$, $(n) \uparrow_k$, and $\binom{n}{k}$ in the same way we used $[n]_q$ to define the q-analogues of these numbers. In general, given the (p, q)-analogue of some formula $\mathcal{F}$, say $\mathcal{F}(p, q)$, if we take $p = 1$, we will get $\mathcal{F}(1, q)$, which is the q-analogue of $\mathcal{F}$.

1.5.1 (P, Q)-Stirling Numbers

As in the previous sections, we can define (p, q)-Stirling numbers of both the first and second kind, and multiple types of these numbers have been defined in previous works (see Hsu-Shieu [16], Médicis-Leroux [19], Remmel-Wachs [24], Wachs-White [28], and Wachs [29]). For this section, however, we will only define one type of (p, q)-Stirling numbers as follows. Define the numbers $S_{n,k}(p, q)$ using the recursions

$$S_{0,0}(p, q) = 1 \text{ and } S_{n,k}(p, q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.70}$$

$$S_{n+1,k}(p, q) = q^k S_{n,k-1}(p, q) + [k]_{p,q} S_{n,k}(p, q) \text{ if } 0 \leq k \leq n + 1 \text{ and } n \geq 0.$$

We call these numbers the *(p, q)-Stirling numbers of the second kind*.

We define the *signless (p, q)-Stirling numbers of the first kind* by the recursions

$$c_{0,0}(p,q) = 1 \text{ and } c_{n,k}(p,q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.71}$$
$$c_{n+1,k}(p,q) = q^n c_{n,k-1}(p,q) + [n]_{p,q} c_{n,k}(p,q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

We then define the (p,q)-*Stirling numbers of the first kind* to be the numbers $s_{n,k}(p,q) = (-1)^{n-k} c_{n,k}(p,q)$, which clearly satisfy the recursions

$$s_{0,0}(p,q) = 1 \text{ and } s_{n,k}(p,q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.72}$$
$$s_{n+1,k}(p,q) = q^n s_{n,k-1}(p,q) - [n]_{p,q} s_{n,k}(p,q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

Given a file placement of rooks, $\mathbb{P}$, in a rook board B, define the following statistics:

1. $\alpha_B(\mathbb{P})$ = the number of cells above a rook in $\mathbb{P}$,
2. $\beta_B(\mathbb{P})$ = the number of cells below a rook in $\mathbb{P}$, and
3. $\epsilon_B(\mathbb{P})$ = the number of cells of B which lie an a column with no rook and are not cancelled by a rook to its left.

We then define the k^{th} (p,q)-*file number of* B to be

$$f_k(B,p,q) = \sum_{\mathbb{P} \in \mathcal{F}_k(B)} q^{\alpha_B(\mathbb{P}) + \epsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P})} \tag{1.73}$$

Using this above definition, we can prove the Equation (1.74) for any board $B = F(b_1, b_2, \ldots, b_n)$:

$$\prod_{i=1}^{n} [x + b_i]_{p,q} = \sum_{k=0}^{n} f_{n-k}(B,p,q)([x]_{p,q})^k \tag{1.74}$$

Now, in much the same way as we proved Theorem 1.5, we can prove the following theorem.

Theorem 1.10. *For all $n \geq 1$ and $0 \leq k \leq n$,*

$$c_{n,k}(p,q) = f_{n-k}(\boldsymbol{B}_n, p, q). \tag{1.75}$$

Combining Theorem 1.10 with (1.74), we obtain the following product formula for the $c_{n,k}(p,q)$:

$$[x]_{p,q} \uparrow_n = \sum_{k=0}^{n} c_{n,k}(p,q)([x]_{p,q})^k. \tag{1.76}$$

By replacing x by $-x$ in (1.76) and multiplying both sides by -1, we can get the following product formula for the $s_{n,k}(p,q)$:

$$[x]_{p,q}\downarrow_n = \sum_{k=0}^{n} s_{n,k}(p,q)([x]_{p,q})^k. \tag{1.77}$$

In the next section, we shall define the p,q-analogues of rook numbers in the Remmel-Wachs j-attacking model. In the special case when $j = 1$, we will obtain the definition of the rook number $r_k(B,p,q)$ for a Ferrers board B. In the special case when $B = \mathbf{B}_n = F(0,1,\ldots,n-1)$, one obtains a second p,q-analogue of the Stirling numbers of the second which are defined by the following recursions:

$$\tilde{S}_{0,0}(p,q) = 1 \text{ and } \tilde{S}_{n,k}(p,q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \tag{1.78}$$
$$\tilde{S}_{n+1,k}(p,q) = q^{k-1}\tilde{S}_{n,k-1}(p,q) + p^{-(n+1)}[k]_{p,q}\tilde{S}_{n,k}(p,q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

That is, the results of Remmel and Wachs imply the following analogue Theorem 1.5.

Theorem 1.11. *For all $n \geq 1$ and $0 \leq k \leq n$,*

$$S_{n,k}(p,q) = r_{n-k}(\boldsymbol{B}_n, p, q). \tag{1.79}$$

Also, the special case to the Remmel-Wachs product formula, see Theorem 1.12, when $j = 1$ and $B = F(0,1,\ldots,n-1)$ will give the following product formula for the $\tilde{S}_{n,k}(p,q)$:

$$([x]_{p,q})^n = \sum_{k=0}^{n} \tilde{S}_{n,k}(p,q)[x]_{p,q}\downarrow_k. \tag{1.80}$$

1.6 Some Generalized Product Formulas $\&$ Their Q-Analogues

Since the original rook model by Goldman, Joichi, and White [11], many other rook models have been developed. A number of mathematicians, by defining an appropriate generalization of the notion of a rook number, have generated whole new classes of product formulas.

1.6.1 The Remmel-Wachs Model

The first model that we will discuss is the *j-Attacking Model* by Remmel and Wachs [24]. Given a fixed $j \in \mathbb{N}$ with $j \geq 1$, we say that a Ferrers board $B(a_1, \ldots, a_n)$ is a *j-attacking* board if for all $1 \leq i < n$, $a_i \neq 0$ implies $a_{i+1} \geq a_i + j - 1$. Suppose that $B(a_1, \ldots, a_n)$ is a j-attacking board and $\mathbb{P}$ is a placement of rooks in $B(a_1, \ldots, a_n)$ which has at most one rook in each column of $B(a_1, \ldots, a_n)$. Then for any individual rook $r \in \mathbb{P}$, we say that r *j-attacks* cell $c \in B(a_1, \ldots, a_n)$ if c lies in a column which is strictly to the right of the column of r and c lies in the first j rows which are weakly above the row of r and which are not j-attacked by any rook which lies in a column that is strictly to the left of r.

For example, suppose $j = 2$ and $\mathbb{P}$ is the placement in $B(1, 2, 3, 5, 7, 8, 10)$ pictured in Figure 1.5. Here the rooks are indicated by placing an $''X''$ in each cell that contains a rook. We place a 2 in each cell attacked by the rook r_2 in column 2. In this case, since there are no rooks to the left of r_2, the cells c which are 2-attacked by r_2 lie in the first two rows which are weakly above the row of r_2, i.e., all the cells in rows 2 and 3 that are in columns 3,4,5,6, and 7. Next consider the rook r_4 which lies in column 4. Again we place a 4 in each of the cells that are 2-attacked by r_4. In this case, the first two rows which lie weakly above r_4 that are not 2-attacked by any rook to the left of r_4 are rows 1 and 4. Thus r_4 2-attacks all the cells in rows 1 and 4 that lie in columns 5, 6, and 7. Finally the rook r_6, which lies in column 6, 2-attacks the cells (6,7) and (7,7) and we place a 6 in these cells. We say that a placement $\mathbb{P}$ is *j-non-attacking* if no rook in $\mathbb{P}$ is j-attacked by a rook to its left and there is at most one rook in each row and column.

Figure 1.5: An example of cells that are 2-attacked in the board $B = F(1,2,3,5,7,8,10)$.

Note that the condition that $B(a_1,\ldots,a_n)$ is j-attacking ensures that any placement $\mathbb{P}$ of j-non-attacking rooks in $B(a_1,\ldots,a_n)$, with at most one rook in each column, has the property that, for any rook $r \in \mathbb{P}$ which lies in a column $k < n$, there are j rows which lie weakly above r and which have no cells which are j-attacked by a rook to the left of r, namely, the row of r plus the top $j-1$ rows in column $k+1$ since $a_{k+1} \geq a_k + j - 1$.

Given a j-attacking board $B = B(a_1,\ldots,a_n)$, we let $\mathcal{N}_k^j(B)$ be the set of all placements $\mathbb{P}$ of k j-non-attacking rooks in B. For example, if $j = 2$ and $B = B(0,2,3,4)$, then $|\mathcal{N}_1^2(B)| = 9$ since there are 9 cells in B, $|\mathcal{N}_2^2(B)| = 6$ and these 6 placements are pictured in Figure 1.6, and $|\mathcal{N}_3^2(B)| = 0$ since any placement $\mathbb{P}$ which has one rook in each nonempty column of B and at most one rook in each row has the property that the rooks in columns 2 and 3 would 2-attack 4 cells in column 4 and hence there would be no place to put a rook in column 4 that is not 2-attacked by a rook to its left. We then define the *k-th j-rook number* of B, $r_k^j(B)$, by setting $r_k^j(B) = |\mathcal{N}_k^j(B)|$.

For any board $B(a_1,\ldots,a_n)$, we let $\mathcal{F}_k(B)$ denote the set of all placements of k rooks in B such that there is at most one rook in each column. We then define

Figure 1.6: The 6 placements in $\mathcal{N}_2^2(B(0,2,3,4))$.

the *k-th file number* of B, $f_k(B)$, to be $f_k(B) = |\mathcal{F}_k(B)|$.

The (P,Q)-Analogue of the Product Formula in the Remmel-Wachs Model

Let $B = B(a_1, \ldots, a_n)$ be a j-attacking board. Then for any placement $\mathbb{P} \in \mathcal{N}_k^j(B)$, we define

$$\tilde{W}^j_{p,q,B}(\mathbb{P}) = q^{a_B(\mathbb{P})} p^{b_B(\mathbb{P})} q^{e_B(\mathbb{P})} p^{-(c_1+\cdots+c_n)j} \tag{1.81}$$

where

1. $a_B(\mathbb{P})$ equals the number of cells of B which lie above a rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$,
2. $b_B(\mathbb{P})$ equals the number of cells of B which lie below a rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$,
3. $e_B(\mathbb{P})$ equals the number of cells of B which lie in a column with no rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$, and
4. $c_1 < \cdots < c_k$ are the columns which contain rooks in $\mathbb{P}$ where we label the columns of B with $1, \ldots, n$ reading from left to right.

For example, in Figure 1.7, we have pictured a placement $\mathbb{P} \in \mathcal{N}_3^3(B)$ where B is the 3-attacking board $B(2,5,8,10,12)$ such that $\mathbb{P}$ has rooks in columns 1, 3 and 4 and $a_B(\mathbb{P}) = 3$, $b_B(\mathbb{P}) = 5$, $e_B(\mathbb{P}) = 5$. Thus $\tilde{W}^3_{p,q,B}(\mathbb{P}) = q^3p^5q^5p^{-(1+3+4)3} = q^8p^{-19}$. Moreover, we have placed a p in each cell of B which contributes to the $b_B(\mathbb{P})$, a q in each cell that contributes to either $a_B(\mathbb{P})$ or $e_B(\mathbb{P})$, and a dot in each cell that is j-attacked by some rook in $\mathbb{P}$.

Figure 1.7: An example of $\tilde{W}_{p,q,B}(\mathbb{P})$

We then define the (p,q)-rook number of B (of type II) by

$$\tilde{r}^{j}_{k,B}(p,q) = \sum_{\mathbb{P}\in\mathcal{N}^{j}_{k}(B)} \tilde{W}^{j}_{p,q,B}(\mathbb{P}). \tag{1.82}$$

Remmel and Wachs [24] proved the following (p,q)-extension of Theorem 1.1

Theorem 1.12. *Let $B = B(a_1,\ldots,a_n)$ be a j-attacking board. Then*

$$\prod_{i=1}^{n}[x + a_i - j(i-1)]_{p,q} = \sum_{k=0}^{n} \tilde{r}^{j}_{k,B}(p,q)p^{kx+\binom{k+1}{2}j}[x]_{p,q}\downarrow_{n-k,j} \tag{1.83}$$

where $[x]_{p,q}\downarrow_{0,j}= 1$ and for $k > 0$, $[x]_{p,q}\downarrow_{k,j}= [x]_{p,q}[x-j]_{p,q}\cdots[x-(k-1)j]_{p,q}$.

Proof Sketch: It is enough to prove (1.83) for all positive integers $x \geq jn$. So fix a positive integer $x \geq jn$ and let B_x be the board which results by adding x rows of length n below B as described in Section 1.2. We shall consider placements of n rooks in B_x where there is at most one rook in each row and column. A rook r which lies above the bar will j-attack cells as described in Section 1.2. Thus a rook r which lies above the bar will only j-attack cells which are above the bar. Similarly, we shall define the cells which a rook r' below the bar j-attacks so that each rook r' will only j-attack cells below the bar in B_x. We say that a rook r' which lies in column k and row l, where here we label the rows below the bar with $1,\ldots,x$ reading from top to bottom, j-attacks a cell $c \in B_x$ which is below the bar only if c lies in a column that is strictly to the right of column k and either

(i) c lies in the first j rows of B_x below the bar which are weakly above row l and which contain no cell that is j-attacked by some rook r'' to the left of r' or

(ii) there are $t < j$ rows below the bar which are weakly above row l and which contain no cell that is j-attacked by some rook r'' which is strictly to the left of column k and c is in the largest $j - t$ rows which are not j-attacked by any rook r'' which is strictly to the left of r'.

In other words, a rook in column k and row l below the bar j-attacks all cells below the bar which are not j-attacked by any rook r'' to the left of r', which are in a column strictly to the right of k and which lie in the first j such rows where we order the rows in the order $l, l-1, \ldots, 1, x, x-1, \ldots, l+1$. Thus when we look for rows for r' to j-attack, we only consider rows below the bar which are not j-attacked by any rook r'' to the left of r'. Then we first look at such rows which are weakly above l, but if there are not j such rows weakly above row l, then we cycle around starting at the bottom row until we find a total of j rows to attack. We then let $\mathcal{N}_k^j(B_x)$ denote the set of all placements $\mathbb{P}$ of n rooks in B_x such that there is at most one rook in each row and column and such that no rook j-attacks another rook. This given, we can then define $W^j_{p,q,B_x}(\mathbb{P})$ just as we did $\tilde{W}^j_{p,q,B}$, namely,

$$W^j_{p,q,B_x}(\mathbb{P}) = q^{a_B(\mathbb{P})} p^{b_B(\mathbb{P})} \tag{1.84}$$

where

1. $a_B(\mathbb{P})$ equals the number of cells of B which lie above a rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$, and
2. $b_B(\mathbb{P})$ equals the number of cells of B which lie below a rook in $\mathbb{P}$ and which are not j-attacked by any rook in $\mathbb{P}$.

For example, consider the placement $\mathbb{P} \in \mathcal{N}_4^3(B_x)$ with $B = F(1, 3, 5, 7)$ and $x = 10$, pictured in Figure 1.8. We shall denote the positions of the four rooks, reading from left to right, by placing circled elements containing the numbers

1, 2, 3 and 4. We shall then indicate the cells which are 3-attacked by the circled rook with label i by placing i's in such cells. We shall place a q or a p in those cells which are not 3-attacked by any rook in $\mathbb{P}$ depending on whether the cell contributes a factor of q or p to $W^j_{p,q,B_x}(\mathbb{P})$ from which it will be clear that $W^j_{p,q,B_x}(\mathbb{P}) = q^8p^{25}$.

Figure 1.8: An example of $\tilde{W}^j_{p,q,B}(\mathbb{P})$

This given, Remmel and Wachs showed that (1.83) results from two different ways of computing the sum

$$S = \sum_{\mathbb{P}\in\mathcal{N}^j_n(B_x)} W^j_{p,q,B_x}(\mathbb{P}) \tag{1.85}$$

by showing that the left-hand side of (1.83) is the result of counting the weights of the placements column by column and the right-hand side of (1.83) is the result of organizing the weights of the placements by the set of rooks which fall in the board B. □

When we talk of the q-analogue of the Remmel-Wachs models, we mean to take the q-statistic on placement of j-non-attacking rooks which results by setting $p = 1$ in the (p, q)-statistic $W^j_{p,q,B}(\mathbb{P})$ or $W^j_{p,q,B_x}(\mathbb{P})$.

1.6.2 The Briggs-Remmel Model

In this section, we describe a variation of the Remmel-Wachs model that is appropriate for rook placements that correspond to partial permutations of the wreath product of the cyclic group of order m, C_m, with the symmetric group $\mathscr{S}_n$, denoted by $C_m \wr \mathscr{S}_n$.

If $\omega = e^{\frac{2\pi i}{m}}$, then we say that $C_m \wr \mathscr{S}_n$ is the group of $m^n n!$ signed permutations where there are m signs, $1 = \omega^0, \omega, \omega^2, \ldots, \omega^{m-1}$. We will usually write the signed permutations in either one-line notation or in disjoint cycle form. For example, if $\sigma \in C_3 \wr \mathscr{S}_8$ is the map with $1 \to \omega 5$, $2 \to 8$, $3 \to \omega^2 3$, $4 \to \omega^2 1$, $5 \to 4$, $6 \to \omega^2 7$, $7 \to \omega 2$, and $8 \to \omega 6$, then in one-line notation,

$$\sigma = \omega 5 \;\; 8 \;\; \omega^2 3 \;\; \omega^2 1 \;\; 4 \;\; \omega^2 7 \;\; \omega 2 \;\; \omega 6,$$

whereas in disjoint cycle form,

$$\sigma = (\omega^2 1 \;\; \omega 5 \;\; 4)(\omega 2 \;\; 8 \;\; \omega 6 \;\; \omega^2 7)(\omega^2 3).$$

That is, in disjoint cycle form, to determine where i is being mapped, we ignore the sign on i and only consider the sign on the element to which it is mapped.

More precisely, we can think of $C_m \wr \mathscr{S}_n$ as a Coxeter-like group with respect to the generating set $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and the relations

$$\begin{aligned} \sigma_i^2 &= 1, \;\; i = 1, \ldots, n-1, \\ \sigma_0^m &= 1, \\ (\sigma_i \sigma_{i+1})^3 &= 1, \;\; i = 1, \ldots, n-2, \\ (\sigma_i \sigma_j)^2 &= 1, \;\; |i-j| \geq 2, and \\ (\sigma_0 \sigma_{n-1})^{2m} &= 1, \end{aligned}$$

where the generators in one line notation are $\sigma_0 = \omega 1 \;\; 2 \;\; 3 \ldots n$, and

$$\sigma_i = 1 \;\; 2 \;\ldots\; i{-}1 \;\; i{+}1 \;\; i \;\; i{+}2 \;\ldots\; n, \qquad \text{for } i = 1, \ldots, n-1.$$

That is, σ_0 sends 1 to $\omega 1$, and σ_i is the transposition of i and $i+1$ for $i = 1, \ldots, n$. It is a well-known fact that the set $\{\sigma_i : 1 \leq i < n\}$ generates the symmetric group $\mathscr{S}_n$.

Given $\sigma \in C_m \wr \mathscr{S}_n$ we will write $\sigma(i)$ as $\varepsilon_i \sigma_i$ where $\sigma_i \in [n] = \{1, \ldots, n\}$ and where $\varepsilon_i = sgn(\sigma_i) \in \{1, \omega, \omega^2, \ldots, \omega^{m-1}\}$ is called the *sign* of σ_i. For each $1 \leq i \leq n$, we define $|\varepsilon_i \sigma_i| = \sigma_i$ and call this the *absolute value* of $\sigma(i)$.

For $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers, let $\omega^j \mathbb{N}$ denote the set $\{\omega^j 1, \omega^j 2, \omega^j 3, \ldots\}$ for $0 \leq j \leq m-1$, and define a total ordering $<_\Theta$ on the set of

Figure 1.9: $B_{n\times 3n}$.

elements in $\mathbb{N}\oplus\omega\mathbb{N}\oplus\omega^2\mathbb{N}\oplus\cdots\oplus\omega^{m-1}\mathbb{N}$, by

$$\omega^{m-1}1<_\Theta\omega^{m-1}2<_\Theta\cdots<_\Theta\omega^{m-2}1<_\Theta\omega^{m-2}2<_\Theta\cdots$$
$$\cdots<_\Theta\omega1<_\Theta\omega2<_\Theta\cdots<_\Theta 1<_\Theta 2<_\Theta\cdots.$$

Adin and Roichman [1] defined the following statistics for permutations in $C_m\wr\mathscr{S}_n$ where $m>2$. For $\sigma=\varepsilon_1\sigma_1\ \varepsilon_2\sigma_2\ldots\varepsilon_n\sigma_n\in C_m\wr\mathscr{S}_n$, they set $Des_m(\sigma)=\{i\in[n-1]:\varepsilon_i\sigma_i>_\Theta\varepsilon_{i+1}\sigma_{i+1}\}$ and for each $1\le j\le m-1$, $N_j(\sigma)=\{i\in[n]:\varepsilon_i=\omega^j\}$. With $n_j(\sigma)=|N_j(\sigma)|$, the number of *descents* and the *flag-major index* of $\sigma\in C_m\wr\mathscr{S}_n$ are respectively defined by

$$des_m(\sigma) = |Des_m(\sigma)| \text{ and} \tag{1.86}$$

$$maj_m(\sigma) = m\sum_{i\in Des_m(\sigma)} i+\sum_{j=1}^{m-1} jn_j(\sigma). \tag{1.87}$$

In addition, we define here the *flag-comajor index* by

$$comaj_m(\sigma)=mn\cdot des_m(\sigma)-maj_m(\sigma).$$

As an example, consider $\sigma=\omega5\ \ 8\ \ \omega^23\ \ \omega^21\ \ 4\ \ \omega^27\ \ \omega2\ \ \omega6\in C_3\wr\mathscr{S}_8$, for which $Des_3(\sigma)=\{2,3,5\}$, $des_3(\sigma)=3$, $maj_3(\sigma)=39$, and $comaj_m(\sigma)=33$.

Let $B_{n\times mn}$ be the $n\times mn$ array of squares where the n columns are labeled from left to right by $1,2,\cdots,n$, and the mn rows are labeled from bottom to top by 1, $\omega1$, $\cdots$, $\omega^{m-1}1$, 2, $\omega2$, $\cdots$, $\omega^{m-1}2$, $\cdots$, n, ωn, $\cdots$, $\omega^{m-1}n$. For instance, the board $B_{n\times 3n}$ is illustrated in Figure 1.9. We let (i,ω^rj) identify the square in the column labeled with i and the row labeled with ω^rj. Each such square will be called a *cell* and the rows labeled by $j,\omega j,\cdots,\omega^{m-1}j$ will be called *level* j.

A *board* will be a subset of cells in $B_{n\times mn}$. In particular, a *skyline board* in $B_{n\times mn}$ is a board whose column heights from left to right are h_1, …, h_n, and is

Figure 1.10: $B \subseteq B_{3\times 6}$.

denoted by $B_m(h_1,\ldots,h_n)$. That is, for each $1 \leq i \leq n$, if $h_i \neq 0$ and $h_i = a_i m + b_i$ with $0 \leq a_i \leq n-1$ and $1 \leq b_i \leq m$, then the ith column contains all of the cells in the set

$$\big\{(i,\omega^s j) \mid 0 \leq s < m, 1 \leq j \leq a_i\big\} \,\cup\, \big\{(i,\omega^s(a_i+1)) \mid 0 \leq s < b_i\big\}.$$

Further, if $0 \leq h_1 \leq \cdots \leq h_n \leq mn$ and for each $1 \leq i \leq n-1$, if $h_i = a_i m + b_i$ where $1 \leq b_i < m$ then $h_{i+1} \geq (a_i+1)m$, then $B_m(h_1,\ldots,h_n)$ is called an *m-Ferrers board* in $B_{n\times mn}$. We will denote the m-Ferrers board with column heights h_1, …, h_n by $F_m(h_1,\ldots,h_n)$.

Given a board $B \subseteq B_{n\times mn}$, we let $R^m_{k,n}(B)$ denote the set of all k element subsets $\mathbb{P}$ of B such that no two elements lie in the same level or column for nonnegative integers k. Such a subset $\mathbb{P}$ will be called a placement of non-attacking m-rooks in B. The cells in $\mathbb{P}$ are considered to contain m-rooks, so that we call $r^m_{k,n}(B) = |R^m_{k,n}(B)|$ the kth *m-rook number* of B. We note that for any board $B \subseteq B_{n\times mn}$, $r^m_{0,n}(B) = 1$, $r^m_{1,n}(B) = |B|$, and if $k > n$, then $r^m_{k,n}(B) = 0$. For example, consider the board in Figure 1.10. One can easily check that $r^2_{0,3}(B) = 1$, $r^2_{1,3}(B) = 9$, $r^2_{2,3}(B) = 18$, and $r^2_{3,3}(B) = 6$.

Given a permutation $\sigma \in C_m \wr \mathscr{S}_n$, we can identify σ with a placement $\mathbb{P}_\sigma$ of n m-rooks in $B_{n\times mn}$. That is, we let $\mathbb{P}_\sigma = \{(i,\omega^r j) : \sigma(i) = \omega^r j\}$ for $1 \leq i \leq n$. We then define $H^m_{k,n}(B) = \{\mathbb{P}_\sigma : \sigma \in C_m \wr \mathscr{S}_n \text{ and } |\mathbb{P}_\sigma \cap B| = k\}$ and we call $h^m_{k,n} = |H^m_{k,n}(B)|$ the kth *m-hit number* of B relative to $B_{n\times mn}$.

Suppose that $B = F_m(b_1,\ldots,b_n) \subseteq B_{n\times mn}$ is an m-Ferrers board and let $\mathbb{P} \in R^m_{k,n}(B)$. A rook in the cell $ij \in \mathbb{P}$ is said to *m-rook-cancel* those cells in the set

$$\big\{(a,\omega^s j) : i < a \leq n, 0 \leq s < m\big\}.$$

Then, we get the following product formula:

Figure 1.11: $\mathbb{P} \in R^3_{2,4}(B)$.

Theorem 1.13. *Let* $B = F_m(b_1, \ldots, b_n) \subseteq B_{n\times mn}$ *be an* m*-Ferrers board. Then*

$$\prod_{i=1}^{n}(mx + b_i - m(i-1)) = \sum_{k=0}^{n} r^m_{k,n}(B)(mx)\downarrow_{(k,m)}, \tag{1.88}$$

where $(x)\downarrow_{(n,m)} = x(x-m)\cdots(x-(k-1)m)$.

The (P,Q)-Analogue of the Product Formula in the Briggs-Remmel Model

Next we define a (p,q)-analogue of the m-rook numbers and prove a (p,q)-analogue of Theorem 1.13.
We then define the kth p,q,m-rook number of B, denoted $r^m_{k,n}(B,p,q)$, as

$$r^m_{k,n}(B,p,q) = \sum_{\mathbb{P}\in R^m_{k,n}(B)} q^{\alpha_B(\mathbb{P})+\varepsilon_B(\mathbb{P})} p^{\beta_B(\mathbb{P})-m(c_1+\cdots+c_k)},$$

where the rooks of $\mathbb{P}$ lie in columns $c_1 < \ldots < c_k$ and where

1. $\alpha_B(\mathbb{P})$ is the number of cells of B which lie above a rook in $\mathbb{P}$ but are not m-rook-canceled by any other rook in $\mathbb{P}$,
2. $\beta_B(\mathbb{P})$ is the number of cells of B which lie below a rook in $\mathbb{P}$ but are not m-rook-canceled by any other rook in $\mathbb{P}$, and, and
3. $\varepsilon_B(\mathbb{P})$ is the number of cells of B which lie in a column with no rook in $\mathbb{P}$ and are not m-rook-canceled by any rook in $\mathbb{P}$.

For example, if $B = F_3(2,4,6,9) \subseteq B_{3\times 12}$ and $\mathbb{P} \in R^3_{2,4}(B)$ is the placement given in Figure 1.11, then $\alpha_B(\mathbb{P}) = 2$, $\beta_B(\mathbb{P}) = 3$, $\varepsilon_B(\mathbb{P}) = 5$, $c_1 = 2$, and $c_2 = 3$. So, the (p,q)-contribution of $\mathbb{P}$ to $R^3_{2,4}(B,p,q)$ is q^7p^{-12}.

With $[x]_{p,q}\downarrow_{(k,m)}$ denoting $[x]_{p,q}[x-m]_{p,q}\cdots[x-m(k-1)]_{p,q}$, we prove the following (p,q)-analogue of the factorization theorem.

Theorem 1.14. *Let* $B = F_m(b_1, \ldots, b_n) \subseteq B_{n\times mn}$ *be an* m*-Ferrers board. Then*

$$\prod_{i=1}^{n}[mx+b_i-m(i-1)]_{p,q} = \sum_{k=0}^{n} r^m_{k,n}(B,p,q)p^{m\left(xk+\binom{k+1}{2}\right)}[mx]_{p,q}\downarrow_{(k,m)}. \tag{1.89}$$

Proof Sketch: Let $x \geq n$ be a positive integer. One can prove the theorem by summing in two different ways the polynomial

$$N_{B_{m,x}} = \sum_{\mathbb{P}\in R^m_{n,n}(B_{m,x})} q^{\alpha_{B_{m,x}}(\mathbb{P})}p^{\beta_{B_{m,x}}(\mathbb{P})},$$

where each rook (either above or below the bar) m-rook-cancels those cells to the right in its level. □

For each $m \geq 1$, Briggs and Remmel also defined a (p,q)-analogue of the Stirling numbers of second in there model by letting $\tilde{S}^m_{n,k}(p,q) = r^m_{n-k,n}(F^{0,m}_n, p, q)$ where $F^{0,m}_n$ denotes the m-Ferrers board $F_m(0, m, 2m, \ldots, m(n-1))$. These are special case of the generalized (p,q)-Stirling numbers of the second kind defined defined by Remmel-Wachs [24]. Then Briggs and Remmel [3] were able to prove the following (p,q)-analogue of the Frobenius formula.

Theorem 1.15. *For each* $m, n \in \mathbb{N}$*,*

$$\sum_{k=1}^{n}\frac{\tilde{S}^m_{n,k}(p,q)[mk]_{p,q}\downarrow_{(k,m)}p^{m\left(\binom{n-k+1}{2}+k(n-k)\right)}x^k}{\prod_{i=0}^{k}(1-xq^{mi}p^{m(n-i)})}$$
$$= \frac{\sum_{\sigma\in C_m\wr\mathscr{S}_n} q^{maj_m(\sigma)}p^{comaj_m(\sigma)}x^{des_m(\sigma)+1}}{\prod_{i=0}^{n}(1-xq^{mi}p^{m(n-i)})}. \tag{1.90}$$

1.6.3 The Haglund-Remmel Perfect Matching Model

The next model we will discuss is the *Perfect Matching Model* by Haglund and Remmel [14]. In this model the board that we consider is called a *perfect matching*

Figure 1.12: A perfect matching board $\mathbf{B}_{2n}$.

board, denoted by $\mathbf{B}_{2n}$, and it is pictured in Figure 1.12, where the columns are labeled from 2 to $2n$ and the rows are labeled from 1 to $2n-1$.

Let p_m denote the set of perfect matchings in the complete graph, K_n, where we call $m = \{\{i_k, j_k\}_{k=1}^{n}\}$ a perfect matching if $1 \leq i_k < j_k \leq 2n$ for every $1 \leq k \leq n$ and $\{i_u, j_u\} \bigcap \{i_v, j_v\} = \emptyset$ for every $u \neq v$. An example of a perfect matching of K_8 with $m = \{\{1,5\},\{2,3\},\{4,7\},\{6,8\}\}$ can be seen in Figure 1.13. We will define any rook placement in a board $B \subseteq \mathbf{B}_{2n}$ to be a subset of some of the set p_m which lies completely in B. We will set $M_k(B) := \{S \subseteq B : \exists p_m \ni p_m \cap B \; = S \textit{ and } |S| = k\}$, and then we will define the k^{th} *rook number of* B to be $m_k(B) := |M_k(B)|$.

Figure 1.13: A example of a perfect matching of K_8.

The board $B = F(b_1, b_2, \ldots, b_{2n-1}) \subseteq \mathbf{B}_{2n}$ is defined as $B = \{(i, i+j) | 1 \leq j \leq b_j\}$, that is, B has row length, from top to bottom, of $b_1, b_2, \ldots, b_{2n-1}$. If $2n-1 \geq b_1 \geq b_2 \geq \cdots \geq b_{2n-1} \geq 0$ and if $b_i > 0$ implies that $b_i > b_{i+1}$ for all $i = 1, 2, \ldots, 2n-2$, then $B = F(b_1, b_2, \ldots, b_{2n-1})$ is called a *shifted Ferrers board*. An example of the shifted Ferrers board $B = F(6,5,3,1,0,0,0) \subset B_8$ can be seen in Figure 1.14.

Figure 1.14: An example of the shifted Ferrers board $B = F(6,5,3,1,0,0,0) \subset B_8$.

Haglund and Remmel then discuss the notion of a *nearly Ferrers board*. They define a nearly Ferrers board, B, as one in which for every cell $(i,j) \in B$, the cells $\{(s,j) : s < i\} \cup \{(t,i) : t < i\} \subseteq B$. By this description, every shifted Ferrers board is also a nearly Ferrers board. Also, one can construct a nearly Ferrers board in the following manner. First fix an $a \in \mathbb{N}$, and then make a triangular arrangement of cells by letting $B' = \{(s,t) | s < t \leq a\}$. One may then add any columns to the right of B', and that board will be nearly Ferrers, as in Figure 1.15.

Figure 1.15: An example of the nearly Ferrers board $B \subset \mathbf{B}_8$.

As in other models, we wish to produce a product formula from this model, and so first we need to make a new board and discuss rook cancellation in that board. Like in the original model by Goldman, Joichi, and White [11], we wish to extend the board $\mathbf{B}_{2n}$ by adding an x-part, which in this case, will be added to the right of the board $\mathbf{B}_{2n}$. We will call this board $\mathbf{B}_{2n,x}$, seen in Figure 1.16.

Figure 1.16: The extended perfect matching board $\mathbf{B}_{2n,x}$.

We now need to define a cancellation on this board, and we will do so in the following way:

1. A rook, r, placed inside the board $\mathbf{B}_{2n}$ in cell (i, j) will cancel the cells $\{(i, s) : i + 1 \leq s \leq 2n + x, s \neq j\}$, $\{(a, i\} : a < i\}$, $\{(b, j) : 1 \leq b \leq j - 1, b \neq i\}$, and $\{(j, t) : t > j\}$.
2. A rook, r, placed in the x-part of the board $\mathbf{B}_{2n,x}$ in the cell (i, j) will cancel all of the cells in its column other than (i, j) as well as all of the cells in the first column to its right in $\mathbf{B}_{2n,x} - \mathbf{B}_{2n}$. If uncancelled columns lie to the right of the j^{th} column in $\mathbf{B}_{2n,x} - \mathbf{B}_{2n}$, then r cancels the cells of column m where m is the largest index of any uncancelled column in $\mathbf{B}_{2n,x} - \mathbf{B}_{2n}$.

An illustration of this kind of cancellation can be seen in Figure 1.17, where the rook labeled as $''X_i''$ cancels the cells marked with an $''i''$. Using this cancellation described above along with the assumption that $x \geq 4n - 2$, Haglund and Remmel [] were able to prove the following product formula:

Figure 1.17: An example of the rook cancellation in the extended perfect matching board $\mathbf{B}_{2n,x}$.

Theorem 1.16. *Let* $B = F(b_1, b_2, \ldots, b_{2n-1})$ *be a nearly Ferrers board* $B \subseteq \mathbf{B}_{2n}$*, and let* b_i *denote the number of cells of* B *that lie in row* i *for each* $1 \leq i \leq 2n-1$*. Then*

$$\prod_{i=1}^{2n-1} (x + b_{2n-i} - 2i + 2) = \sum_{k=0}^{2n-1} m_{n-k}(B)(x) \downarrow\downarrow_{n+k-1}, \tag{1.91}$$

where $(x) \downarrow\downarrow_k = x(x-2)(x-4)\cdots(x-2(k-1))$.

The Q-Analogue of the Haglund-Remmel Model

Suppose that we are given a nearly Ferrers board $B = F(b_1, b_2, \ldots, b_{2n-1})$ and a placement $\mathbb{P} \in M_k(B)$. We define the k^{th} q-rook number of B to be

$$m_k(B, q) = \sum_{\mathbb{P} \in M_k(B)} q^{w_B(\mathbb{P})}, \tag{1.92}$$

where $w_B(\mathbb{P})$ is the number of cells in $B - \mathbb{P}$ that are not rook cancelled by any of the rooks in $\mathbb{P}$, and where we set $m_0(B, q) = q^{|B|}$.

Theorem 1.17. *Let* $B = F(b_1, b_2, \ldots, b_{2n-1})$ *be a nearly Ferrers board* $B \subseteq \mathbf{B}_{2n}$*, and let* b_i *denote the number of cells of* B *that lie in row* i *for each* $1 \leq i \leq 2n-1$*. Then*

$$\prod_{i=1}^{2n-1} [x + b_{2n-i} - 2i + 2]_q = \sum_{k=0}^{2n-1} m_{n-k}(B, q)[x]_q \downarrow\downarrow_{2n+k-1} \tag{1.93}$$

Proof Sketch: □

1.6.4 The Goldman-Haglund Generalized Rook Model

The i-Creation Model

A model which produces a product formula which has rising factorials on the right-hand side is the *i-Creation Model* due to Goldman and Haglund [10].

Figure 1.18: A placement of 3 i-creation rooks in $B^{(i)}$ where $B = F(1,2,2,4,4)$ and $i = 2$.

In this model, new cells are created after an *i-creation rook* is placed in the board, rather than cells being cancelled. For $i \in \mathbb{N}$, we call $B^{(i)} = F(b_1, b_2, \dots, b_n)$ an *i-creation board* if $B = F(b_1, b_2, \dots, b_n)$ is a Ferrers board, and, when an i-creation rook is placed in $B^{(i)}$, it replaces all the cells in its row to its right with $i+1$ cells, the lowest of which get cancelled - a process called *i-creation*. The next i-creation rook may then be placed in any available cell, both those that were part of the original board and those that have been *i-created*. An example of a 2-creation board and a placement of 3 i-creation rooks can be seen in Figure 1.18.

We will denote the set of placements of k rook in an i-creation board, $B^{(i)}$, by $\mathcal{F}_k^{(i)}(B)$. Goldman and Haglund denote the number of ways of placing k rooks in $B^{(i)}$ by $r_k^{(i)}(B) = |\mathcal{F}_k^{(i)}(B)|$, the k^{th} *i-creation rook number of B*. We may also note here that for the i creation board $B^{(i)}$ with $B = F(b_1, b_2, \dots, b_{n+1})$ these i-creation rook numbers satisfy the recursion

$$r_{n+1-k}^{(i)}(b_1, b_2, \dots, b_{n+1}) = r_{n+1-k}^{(i)}(b_1, b_2, \dots, b_n) + (b_{n+1} + k(i-1)) r_{n-k}^{(i)}(b_1, b_2, \dots, b_n).$$

Moreover, for any Ferrers board B, $r_k^{(0)}(B) = r_k(B)$ and $r_k^{(1)}(B) = f_k(B)$.

The board $B_x^{(i)}$ is defined to be the board $B^{(i)}$ with an x-part appended below, and rooks placed in the x-part of $B_x^{(i)}$ will i-create and cancel exactly as would an i-creation rook placed in $B^{(i)}$. Then Haglund and Goldman [10] proved the following product formula.

Theorem 1.18. *Let $B^{(i)} = F(b_1, b_2, \dots, b_n)$ be an i-creation board for some $i \in \mathbb{N}$. For all $x \in \mathbb{N}$,*

$$\prod_{j=1}^{n} (x + b_j + (j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(i)}(B) x^{(k,i-1)}, \tag{1.94}$$

where $x^{(n,m)} = x(x+m)\cdots(x+(n-1)m)$ and $x^{(0,m)} = 1$.

The α-Parameter

A more general rook placement setting was also defined by Goldman and Haglund in [10]. Here, given a Ferrers board B, we consider placements $\mathbb{P} \in \mathcal{F}_k(B)$. Given a placement $\mathbb{P} \in \mathcal{F}_k(B)$, we define the weight of $\mathbb{P}$, $w(\mathbb{P})$, to be the product of the weights of all of the rows of the placement, where if a row r contains u rooks, then it has weight

$$w(r) = \begin{cases} 1 & \text{if } 0 \le u \le 1, \text{ and} \\ \alpha(2\alpha-1)(3\alpha-2)\cdots((u-1)\alpha-(u-2)) & \text{if } u \ge 2. \end{cases}$$

We then set

$$r_k^{(\alpha)}(B) = \sum_{\mathbb{P} \in \mathcal{F}_k(B)} w(\mathbb{P}),$$

and we call $r_k^{(\alpha)}(B)$ the k^{th} *α-rook number of B*. Haglund and Goldman [] proved the following theorem.

Theorem 1.19. *If $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board and α is an integer, then*

$$\prod_{j-1}^{n}(x + b_i + (j-1)(\alpha-1)) = \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B) x^{(k,\alpha-1)}. \tag{1.95}$$

Moreover, we may note here that if $\alpha \in \mathbb{N}$, then $r_k^{(\alpha)}(B)$ is the α-creation rook number just described. If $\alpha \in \mathbb{Z}^-$, then for a suitable board, $r_k^{(\alpha)}(B)$ is a j-attacking rook as defined by the the model of Remmel and Wachs [24].

The Q-Analogue of the Goldman-Haglund Model

Suppose that $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board and consider $\mathbb{P} \in \mathcal{F}_k(B)$. Let c be any cell of B and define $\nu(c)$ to be the number of rooks which are strictly to the left of, and in the same row as, c. We then define the weight of c, denoted by $w_B(c)$, to be

$$w_B(c) = \begin{cases} 1 & \text{if there is a rook directly above } c \\ [(\alpha-1)\nu(c)+1]_q & \text{if there is a rook in } c\text{, and} \\ q^{(\alpha-1)\nu(c)+1} & \text{otherwise,} \end{cases}$$

and then define the placement weight, $w_B(\mathbb{P})$ to be

$$w_B(\mathbb{P}) = \prod_{c \in B} w_B(c). \tag{1.96}$$

Then we define the k^{th} *$q\alpha$-rook number*, $r_k^{(\alpha)}(B,q)$ by

$$r_k^{(\alpha)}(B,q) = \sum \mathbb{P} \in \mathcal{F}_k(B) w_B(\mathbb{P}). \tag{1.97}$$

We now can prove the following q-analogue formula:

Theorem 1.20. *If $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board, then*

$$\prod_{i=1}^{n} [x + b_i - (j-1)(\alpha-1)]_q = \sum_{k=0}^{n} r_{n-k}^{(\alpha)}(B,q)[x^{(k,\alpha-1)}]_q, \tag{1.98}$$

where $[x^{(n,m)}]_q = [x]_q[x+m-1]_q \cdots [x+(n-1)(m-1)]_q$.

Proof Sketch: This is proved in the standard way, of equating the q-weighting of all rook placements in B_x in two ways: placing rooks in the columns from left to right or first fixing a placement of $n-k$ rooks above the bar and extending that to a placement in the entire board. □

Chapter 2

Product Formulas $\&$ General Rook Boards

2.1 Introduction

In the previous chapter, we considered various models in rook theory that lead to product formulas. For example, the original model of Goldman, Joichi, and White model [11] and subsequent generalizations of that model due to Remmel and Wachs [24], Briggs and Remmel [4], and Haglund and Remmel [14] led to product formulas of the form

$$\prod_{i=1}^{n}(x + b_i - y_{i,j}) = \sum_{k=0}^{n} r_{n-k}(B)x(x-j)\cdots(x-j(k-1)) \tag{2.1}$$

for appropriate j and $y_{i,j}$. The Haglund and Goldman [10] j-creation model led to product formulas of the form

$$\prod_{i=1}^{n}(x + b_i + y_{i,j}) = \sum_{k=0}^{n} \tilde{r}_{n-k}(B)x(x+j)\cdots(x+j(k-1)) \tag{2.2}$$

for appropriate j and $y_{i,j}$.

In this chapter, we shall develop a new rook theory model where we can prove much more general product formulas that include all the product formulas described in Chapter 1, more specifically, we will give a combinatorial proof of the following theorem, from which any of the product formulas from Chapter 1 will immediately follow.

Theorem 2.1. *Suppose we are given any two sequences* $\mathcal{B} = \{b_i\}_{i=1}^n$ *and* $\mathcal{A} = \{a_i\}_{i=1}^n$ *with* $\mathcal{B}, \mathcal{A} \in \mathbb{N}^n$ *and two functions* $sgn, \overline{sgn} : [n] \to \{-1, +1\}$. *Then*

$$\prod_{i=1}^{n}(x + sgn(i)(b_i)) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) \prod_{j=1}^{k}(x + \sum_{s \leq j} \overline{sgn}(s)(a_s)). \tag{2.3}$$

We will refer to Equation 2.3 as the *general product formula* and the number $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ as the k^{th} *augmented rook number of* $\mathcal{B}$ *with respect to* $\mathcal{A}$, sgn, *and* $\overline{sgn}$.

2.2 General Augmented Rook Boards

Let $\mathcal{B} = \{b_i\}_{i=1}^n, \mathcal{A} = \{a_i\}_{i=1}^n \in \mathbb{N}^n$ be any sequences. Let $A_i = a_1 + a_2 + \cdots + a_i$ be the i^{th} partial sum of the a_i's and let $B = F(b_1, b_2, \ldots, b_n)$. We will consider the *augmented rook board,* $\mathcal{B}^{\mathcal{A}}$ which is constructed by starting with the board B and then adding $a_1 + \cdots + a_i$ on top of the i-th column for $i = 1, \ldots, n$. Thus $\mathcal{B}^{\mathcal{A}}$ can be thought of as the board $F(b_1 + A_1, b_2 + A_2, \ldots, b_n + A_n)$. For example, if $\mathcal{B} = (1, 2, 2, 3)$ and $\mathcal{A} = (2, 1, 2, 1)$, then Figure 2.1 pictures the board B and the board $\mathcal{B}^{\mathcal{A}}$. We will refer to the part of the board consisting of the b_i's as the *$\mathcal{B}$-part of $\mathcal{B}^{\mathcal{A}}$* and the part which corresponds to the a_i's as the *augmented part of* $\mathcal{B}^{\mathcal{A}}$. Moreover, for each column i, we will call the cells in rows $b_1 + 1, \ldots, b_1 + a_1$ as that a_1-st part of i-th column, the cells in row $b_1 + a_1 + 1, \ldots, b_1 + a_1 + a_2$ as that a_2-nd part of i-th column, etc. In Figure 2.1, we have indicated the a_s-th part of each column by putting an s in those cells.

Next we must define the appropriate notion of non-attacking rook placements in $\mathcal{B}^{\mathcal{A}}$. We first consider placements $\mathbb{P}$ of rooks in $\mathcal{B}^{\mathcal{A}}$ where there is at

Figure 2.1: An Augmented Rook Board, $\mathcal{B}^{\mathcal{A}}$, with $n = 4$.

Figure 2.2: A Placement of Two Rooks in an Augmented Rook Board, $\mathcal{B}^{\mathcal{A}}$.

most one rook in each column. Now the left most rook of $\mathbb{P}$ will cancel all cells in the columns to its right which correspond to the a_s-th part of that column of highest index. Thus, if the leftmost rook is in column i, then it will cancel the a_j part of column j for $j = i + 1, \ldots, n$. In general, each rook will cancel all cells in the columns to its right which correspond to the a_s-th part of that column where s is the highest index such that the cell of a_s-th part of the column has not been cancelled by any rook to its left. We then let $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ denote the set of placements of k rooks in the board $\mathcal{B}^{\mathcal{A}}$ such that (i) there is at most one rook per column and (ii) no rook lies in cell which has been cancelled by a rook to its left. For example, if $\mathcal{B} = (1, 2, 2, 3)$ and $\mathcal{A} = (2, 1, 2, 1)$, then we have illustrated in Figure 2.2 a placement where we place a "•" in all cells cancelled by rook in column 1 and a "*" in all cells cancelled by the rook in column 2. We shall refer to a placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(B)$ as a placement of k non-attacking rook in $\mathcal{B}^{\mathcal{A}}$.

We wish to define the analogue of the board B_x for augmented rook boards, which we call a *general augmented rook board* and which will be denoted by $\mathcal{B}_x^{\mathcal{A}}$. Given two sequences of nonnegative integers $\mathcal{B}$ and $\mathcal{A}$ and a nonnegative integer x, the board $\mathcal{B}_x^{\mathcal{A}}$ will be defined by three parts. First we start with the board $\mathcal{B}^{\mathcal{A}}$ which will refer to as the *upper part of* $\mathcal{B}_x^{\mathcal{A}}$, where the part of the upper part that corresponds to the board $B = F(b_1, b_2, \ldots, b_n)$ will be called the *$\mathcal{B}$-part* and the part which correspond to the a_i's will be called the *upper augmented part*. Directly below $\mathcal{B}^{\mathcal{A}}$, we will attach x-rows of length n which will be referred to as the *x-part of* $\mathcal{B}_x^{\mathcal{A}}$. Finally, directly below the x-part, we will place the flip of a Ferrers board $F(A_1, \ldots, A_n)$ which will be call the *lower augmented part* of $\mathcal{B}_x^{\mathcal{A}}$. We will say that x-part is separated from the upper part of $\mathcal{B}_x^{\mathcal{A}}$ by the *high bar* and from the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ by the *low bar*. For example, Figure 2.3

Figure 2.3: An Example of an Augmented General Rook Board, $B_x^{\mathcal{A}}$, with $\mathcal{B} = (1,2,2,3)$, $\mathcal{A} = (2,1,2,1)$, and $x = 4$.

pictures the board $\mathcal{B}_x^{\mathcal{A}}$ where $\mathcal{B} = (1,2,2,3)$, $\mathcal{A} = (2,1,2,1)$, and $x = 4$. Much like we did for the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$, we will refer to the first a_1 cells of the lower augmented board of a column i, reading from top to bottom, as the a_1-st part of the i-th column of the lower augmented board, the next a_2 cells, reading from top to bottom, as the a_2-nd part of the i-th column of the lower augmented board, etc. Again, we indicate the a_s-th part of each column by placing an s in those cells.

Next we need to define the set of placements of n non-attacking rooks on the board B_x for the board $\mathcal{B}_x^{\mathcal{A}}$. First we will consider placements of n rooks on $\mathcal{B}_x^{\mathcal{A}}$ where there is exactly one rook in each column. The cancellation rules for each rook are the following:

1. A rook placed above the high bar in the j^{th} column of $\mathcal{B}_x^{\mathcal{A}}$ will cancel all of the cells in columns $j+1, j+2, \ldots, n$, in both the upper and lower augmented parts, which belong to the a_i of highest subscript in that column which has not yet been cancelled by a previously placed rook.

2. Rooks placed below the high bar do not cancel anything.

We then let $\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ denote the set of all placements of n rooks in $\mathcal{B}_x^{\mathcal{A}}$ for which there is exactly one rook in each column and no rook lies in cell which is cancelled by a rook to its left.

An example of a place $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ is in Figure 2.4. Here we have indicated the cells cancelled by the rook in the first column of upper-augmented part by placing a "•" in those cells and the cells cancelled by the rook in the second column of the upper-augmented part by placing an "$*$" in those cells. The rooks placed in the third and fourth columns do not cancel any cells since they are placed below the high bar.

Figure 2.4: An Example of an Augmented General Rook Board, B_x^A, with $\mathcal{B} = (1,2,2,3)$, $\mathcal{A} = (2,1,2,1)$, and $x = 4$, and a placement of rooks in $\mathcal{B}_x^{\mathcal{A}}$.

2.3 The General Product Formula

2.3.1 Two Special Cases of the General Product Formula

In this section, we shall start out by proving two special cases of Theorem 2.1 before we give the general proof. First we consider the case where $sgn(i) = +1$ and $\overline{sgn}(i) = -1$ for every $1 \leq i \leq n$, and in this case we will set

$$r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}).$$

Thus, we want to prove Equation (2.4):

$$\prod_{i=1}^{n}(x + b_i) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})(x - A_1)(x - A_2)\cdots(x - A_k). \tag{2.4}$$

To prove this formula, we will first prove two lemmas.

Lemma 2.2. *For any placement $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$, if there are $b_j + A_m$ uncancelled cells in the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$ in column j, then there are A_m uncancelled cells in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ bar to place a rook in column j.*

Proof: This follows directly from our definition of cancellation for placements $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ since it is easy to see by induction on j that for any $1 \leq s \leq j$, the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$ is cancelled by a rook r to the left of column j if and only if the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ is cancelled by r. □

Lemma 2.3. *For any placement $\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$, if k rooks are placed above the high bar in $\mathcal{B}_x^{\mathcal{A}}$, then the cells which are not cancelled in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ in the i-th column from the left which does not contain a rook above the bar are precisely the cells corresponding the a_s part of that column for $s = 1, \ldots, i$. Thus the column heights of the uncancelled cells in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ in those columns which do not contain rooks above the high bar are $A_1, \ldots, A_{n-k}$, reading from right to left.*

Proof: We proceed by induction on the number of rooks k placed above the high bar in $\mathbb{P}$. Clearly, if $k = 0$, then all the rooks of $\mathbb{P}$ are placed below the high bar. Since our definitions ensure that rooks placed below the high bar do not cancel any cells, the lemma follows in this case from our definition of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Now assume that the lemma holds for some $k \geq 0$ and suppose that $\mathbb{P}$ has $k+1$ rooks above the high bar such that the rightmost of these rooks, r, is placed in column j for some $k+1 \leq j \leq n$. When constructing $\mathbb{P}$, suppose that we first place the first k rooks above the high bar, from left to right. Then, by induction, in the $j-k-1$ columns available below the low bar to the left of column j, there will be, from left to right, $A_1, A_2, \ldots, A_{j-k-1}$ available cells to place a rook in each of those columns. Also from our induction hypothesis, column j will have A_{j-k} available cells, and columns $j+1, j+2, \ldots, n$ will have $A_{j-k+1}, A_{j-k+2}, \ldots, A_{n-k}$ available cells respectively. Now, when we place r in column j above the high bar, then the number of available cells to the left of r below the low bar will remain unchanged and there are no longer any available cells below the low bar in column j since there is now a rook in that column. It is easy to see from our definitions that below the low bar to the right of r, the number of available cells in column $j+a$ for each $a = 1, 2, \ldots, n-j$ will be $A_{j-k+a} - a_{j-k+a} = A_{j-k+a-1}$. Thus, the number of available cells below the low bar in the columns to the right of r are

$$A_{(j-k+1)-1}, A_{(j-k+2)-1}, \ldots, A_{(n-k)-1} = A_{j-k}, A_{j-k+1}, \ldots, A_{n-(k+1)},$$

which completes the induction. $\square$

We are now in a position to prove Equation (2.4).

Proof of Equation (2.4): Let

$$S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) = \sum_{\mathbb{P} \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} w_{sgn,\overline{sgn}}(\mathbb{P}) \tag{2.5}$$

where

$$w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{n} w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) \tag{2.6}$$

where r_i is the rook in the i-th column of $\mathbb{P}$ and

1. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = \overline{sgn}(s)$ if r_i is in the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$,

2. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the x-part of $\mathcal{B}_x^{\mathcal{A}}$,

3. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = sgn(i)$ if r_i is in the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$, and

4. $w_{sgn,\overline{sgn},,\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = -\overline{sgn}(s)$ if r_i is in the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Similarly, we define

$$r_k(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}}) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(B)} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \tag{2.7}$$

where

$$w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{r \in \mathbb{P}} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) \tag{2.8}$$

and, for any rook r, if r is the rook in the i-th column, then

1. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r) = sgn(i)$ if r is in $\mathcal{B}$-part of $\mathcal{B}^{\mathcal{A}}$, and

2. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r) = -\overline{sgn}(s)$ if r is in the a_s-th part of the augmented part of $\mathcal{B}^{\mathcal{A}}$.

Note that in the special case where $sgn(i) = +1$ and $\overline{sgn}(i) = -1$ for every $1 \leq i \leq n$, it is easy to see that

1. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = -1$ if r_i is in the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$,

2. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the x-part of $\mathcal{B}_x^{\mathcal{A}}$,
3. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$, and
4. $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}},\mathbb{P}}(r_i) = 1$ if r_i is in the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Thus, in this case, $w_{sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(\mathbb{P}) = (-1)^{l_{\mathbb{P}}}$ where $l_{\mathbb{P}}$ is the number of rooks in $\mathbb{P}$ which lie in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$. Similarly, $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = |\mathcal{N}_k^{\mathcal{A}}(B)| = r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$.

Then we claim that (2.4) arises from two different ways of computing the sum $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$.

If we first place the rooks starting with the leftmost column and working to the right, then we can see that in the first column there are exactly $x + b_1 + 2a_1$ cells in which to place the first rook, where the "$2a_1$" corresponds to placing the rook in either the upper or lower augmented part of the 1^{st} column. Since all of the rooks above the high bar are weighted with a "$+1$" and all of the rooks placed below the low bar are weighted with a "-1", we get a total weighting of $x + b_1 + a_1 + (-a_1) = x + b_1$ for the first column. When we go to place a rook in the second column, we have two cases.

<u>Case I:</u> Suppose the rook that was placed in the first column was placed below the high bar. Then nothing was cancelled in the second column, so we can place a rook in any cell of the second column. Thus we have $x + b_2 + 2(a_1 + a_2)$ choices as to where to put this rook. However, we weight the two choices which correspond to the "$2(a_1 + a_2)$" term differently, as rooks in the upper augmented part get weighted with a "$+1$" and those in the lower augmented part with a "-1". Thus, the weighting for this column is $x+b_2+(a_1+a_2)+(-a_1-a_2) = x+b_2$.

<u>Case II:</u> If the rook placed in the first column was placed above the high bar, then the cells corresponding to a_2 in both the upper and lower augmented parts of the 2^{nd} column are cancelled, so there are $x + b_2 + 2a_1$ cells left to place the rook, and the weighting is $x + b_2 + a_1 + (-a_1) = x + b_2$.

In general, suppose we are placing a rook in the j^{th} column, and suppose that we have placed s rooks above the high bar and t rooks below the high bar in the first $j-1$ columns. Then in the j^{th} column we have, by Lemma 2.2, $x + b_j + 2(A_{t+1})$ choices as to where to place the rook in that column. Again, these placements will come with a weighting of $x + b_j + A_{t+1} + (-A_{t+1}) = x + b_j$. Thus, it follows that

$$S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) = \prod_{i=1}^{n}(x + b_i)$$

which gives the left-hand side of (2.4).

The second way of counting over the weights of all the rook placements in $\mathcal{B}_x^{\mathcal{A}}$ is to organize the placements by how many rooks lie above the high bar. Suppose that we fix a placement $\mathbb{P}$ of $n-k$ non-attaching rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q \in \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn}}(Q). \tag{2.9}$$

Each such Q in the sum arises from $\mathbb{P}$ by placing rooks below the bar in the remaining columns. Thus there are k columns left that need to have rooks placed in them, below the high bar. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 2.3, the number of ways we can do this will be $(x + A_1)(x + A_2) \cdots (x + A_k)$. However, as all the rooks in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ have weight -1 and all the weights of the rooks in x-part of $\mathcal{B}_x^{\mathcal{A}}$ have weight 1, it is easy to see that

$$\sum_{\substack{Q \in \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn}}(Q) = (x + (-A_1))(x + (-A_2)) \cdots (x + (-A_k)). \tag{2.10}$$

Thus,

$$\begin{aligned} S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} (x - A_1)(x - A_2) \cdots (x - A_k) \\ &= \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})(x - A_1)(x - A_2) \cdots (x - A_k) \end{aligned}$$

which gives the right-hand side of (2.4). □

Next consider the case where $sgn(i) = +1$ and $\overline{sgn}(i) = +1$ for every $1 \leq i \leq n$. In this case, it is easy to see that

1. $w_{sgn,\overline{sgn},\mathbb{P}}(r_i) = 1$ if r_i is in the a_s-th part of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$,
2. $w_{sgn,\overline{sgn},\mathbb{P}}(r_i) = 1$ if r_i is in the x-part of $\mathcal{B}_x^{\mathcal{A}}$,
3. $w_{sgn,\overline{sgn},\mathbb{P}}(r_i) = 1$ if r_i is in the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$, and
4. $w_{sgn,\overline{sgn},\mathbb{P}}(r_i) = -1$ if r_i is in the a_s-th part of the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$.

Thus, in this case, $w_{sgn,\overline{sgn}}(\mathbb{P}) = (-1)^{u_{\mathbb{P}}}$ where $u_{\mathbb{P}}$ is the number of rooks in $\mathbb{P}$ which lie in the upper augmented part of $\mathcal{B}_x^{\mathcal{A}}$. Hence $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(B)} (-1)^{u_{\mathbb{P}}}$.

Again considering our two different ways of computing the sum $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$, we see that if we consider placing the rooks column by column then the sum of the weight of placement of the rook in the i-th column is still $(x + b_i)$ because that argument depended only on the fact that weights of the uncancelled cells in the upper and lower augmented parts of the board in the i-th column cancel. Thus,

$$S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) = \prod_{i=1}^{n} (x + b_i)$$

as before.

The second way of counting over the weights of all the rook placements in $\mathcal{B}_x^{\mathcal{A}}$ is to organize the placements by how many rooks lie above the high bar. Suppose that we fix a placement $\mathbb{P}$ of $n-k$ non-attaching rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q \in \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn}}(Q). \tag{2.11}$$

As before, each such Q in the sum arise from $\mathbb{P}$ by placing rooks below the bar in the remaining columns. Thus there are k columns left that need to have rooks placed in them, below the high bar. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 2.3, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_k)$. Since the weights of all rooks below the bar is 1

$$\sum_{\substack{Q \in \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} w_{sgn,\overline{sgn}}(Q) = w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P})(x+A_1)(x+A_2)\cdots(x+A_k), \tag{2.12}$$

which gives us

$$\begin{aligned} S(sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P})(x+A_1)(x+A_2)\cdots(x+A_k) \\ &= \sum_{k=0}^{n} r_{n-k}(sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}})(x+A_1)(x+A_2)\cdots(x+A_k). \end{aligned}$$

Thus, in this case, we have

$$\prod_{i=1}^{n}(x+b_i) = \sum_{k=0}^{n} \tilde{r}_{n-k}^{\mathcal{A}}(\mathcal{B})(x+A_1)(x+A_2)\cdots(x+A_k) \tag{2.13}$$

where $\tilde{r}_{n-k}^{\mathcal{A}}(\mathcal{B}) = r_{n-k}(+1,-1,\mathcal{B}^{\mathcal{A}})$.

We see that these two special cases encapsulate all the $q=1$ cases of the product formulas stated in the Section 2.1.

2.4 The Proof of the General Product Formula

In this section, we shall prove Theorem 2.1.

Proof: We claim that Equation (2.3) arises from two different ways of computing the sum $S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}})$.

If we first place the rooks starting with the leftmost column and working to the right, then we can see that in the first column there are exactly $x + b_1 + 2a_1$ cells in which to place the first rook, where the "$2a_1$" corresponds to placing the rook in either the upper or lower augmented part of the first column. Since the rooks in the x-part are weighted with a $+1$, the rooks in the i-th column of the $\mathcal{B}$-part are weighted with $sgn(i)$, the rooks placed in the lower augmented part in the a_s-part are weighted with $\overline{sgn}(s)$, and the rooks placed in the a_s-part of the upper augmented part are weighted with $-\overline{sgn}(s)$, we get a total weighting of $x + sgn(1)b_1 + (\overline{sgn}(1)a_1 + (-\overline{sgn}(1)a_1) = x + sgn(1)b_1$ for the first column. When we go to place a rook in the second column, we have two cases.

Case I: Suppose the rook that was placed in the first column was placed below the high bar. Then nothing was cancelled in the second column, so we can place a rook in any cell of the second column. Thus we have $x + b_2 + 2(a_1 + a_2)$ choices as to where to put this rook. However, we weight the two choices which correspond to the "$2(a_1 + a_2)$" term differently, as rooks in the upper augmented part get weighted with $-\overline{sgn}(1)a_1 - \overline{sgn}(2)a_2$ and those in the lower augmented part with $\overline{sgn}(1)a_1 + \overline{sgn}(2)a_2$. Thus, the weighting for this column is $x + sgn(2)b_2 + (\overline{sgn}(1)a_1 + \overline{sgn}(2)a_2) - (\overline{sgn}(1)a_1 + \overline{sgn}(2)a_2) = x + sgn(2)b_2$.

Case II: If the rook placed in the first column was placed above the high bar, then the cells corresponding to a_2 in both the upper and lower augmented parts of the 2^{nd} column are cancelled, so there are $x + b_2 + 2a_1$ cells left to place the rook, and the weighting is $x + sgn(2)b_2 + \overline{sgn}(1)a_1 - \overline{sgn}(1)a_1 = x + sgn(2)b_2$.

In general, suppose we are placing a rook in the j^{th} column, and suppose that we have placed s rooks above the high bar and t rooks below the high bar in the first $j-1$ columns. Then in the j^{th} column we have, by Lemma 2.2, $x+b_j+2(a_1+\cdots+a_{t+1})$ choices as to where to place the rook in that column. Since the weight of cells in the a_s part of the upper augmented board in this column is $-\overline{sgn}(s)$ and the weight of cells in the a_s part of the lower augmented board in this column is $\overline{sgn}(s)$, it follows that these placements will come with a weighting of $x+sgn(j)b_j+\sum_{i=1}^{t+1}(\overline{sgn}(i)a_i-\overline{sgn}(i)a_i)=x+sgn(j)b_j$. Thus, it follows that

$$S(sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}})=\prod_{i=1}^{n}(x+sgn(i)b_i)$$

which gives the lefthand side of (2.3).

The second way of counting over the weights of all the rook placements in $\mathcal{B}_x^{\mathcal{A}}$ is to organize the placements by how many rooks lie above the high bar. Suppose that we fix a placement $\mathbb{P}$ of $n-k$ non-attaching rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q\in\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})\\ Q\cap\mathcal{B}^{\mathcal{A}}=\mathbb{P}}} w_{sgn,\overline{sgn}}(Q). \tag{2.14}$$

Again, each such Q in the sum arise from $\mathbb{P}$ by placing rooks below the high bar in the remaining columns. Thus there are k columns left that need to have rooks placed in them, below the high bar. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 2.3, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_k)$. However, as all the rooks in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ have weight $\overline{sgn}(i)$ if they are in the a_i-th part of the column in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ and all the weights of the rooks in x-part of $\mathcal{B}_x^{\mathcal{A}}$ have weight 1, we see that

$$\sum_{\substack{Q\in\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})\\ Q\cap\mathcal{B}^{\mathcal{A}}=\mathbb{P}}} w_{sgn,\overline{sgn}}(Q)=w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P})\prod_{j=1}^{k}(x+\sum_{s\le j}\overline{sgn}(s)(a_s)). \tag{2.15}$$

Thus,

$$\begin{aligned} S(sgn, \overline{sgn}, \mathcal{B}_x^{\mathcal{A}}) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \prod_{j=1}^{k} (x + \sum_{s \le j} \overline{sgn}(s)(a_s)) \\ &= \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) \prod_{j=1}^{k} (x + \sum_{s \le j} \overline{sgn}(s)(a_s)) \end{aligned}$$

which gives the righthand side of (2.3). □

2.4.1 Comparisons With Other Rook Models

In this section, we shall compare our rook model to the j-attacking rook model of Remmel-Wachs [24] and the j-creation model of Goldman-Haglund [10]. In particular, we want to compare the rook numbers that correspond to a given product formula in our model versus the other models.

The Remmel-Wachs j-Attacking Model

We start with the Remmel-Wachs model. Suppose that we are given a j-attacking board $D = (d_1, \ldots, d_n)$, we let $\mathcal{N}_k^j(D)$ denote the set of all placements of k rooks in B such that there is at most one rook in each column and no rook is in a cell that is j-attacked by a rook to its left. Then in the Remmel-Wachs model, D gives rise to the following product formula:

$$\prod_{i=1}^{n} (x + d_i - j(i-1)) = \sum_{k=0}^{n} \tilde{r}_{n-k,D}^{j} x(x-j)\cdots(x-(k-1)j) \tag{2.16}$$

where $\tilde{r}_{n-k,D}^{j}(1,1) = |\mathcal{N}_k^j(D)|$. Now if we want to obtain the same product in our model, we must start with the sequences $\mathcal{B} = (d_1, |d_2 - j|, \ldots, |d_n - (n-1)j)|$ and $\mathcal{A} = (0, j, j, \ldots, j)$. We also define the sign functions, sgn and $\overline{sgn}$ such that for all $i = 1, \ldots, n$,

Figure 2.5: $F(1,3,6,8)$ versus $\mathcal{B}^{\mathcal{A}}$ where $\mathcal{B} = (1,1,2,2)$ and $\mathcal{A} = (0,2,4,6)$.

$$\begin{aligned}
\overline{sgn}(i) &= 1 \text{ and} \\
sgn(i) &= \begin{cases} 1 & \text{if } d_i - j(i-1) \geq 0 \\ -1 & \text{if } d_i - j(i-1) < 0. \end{cases}
\end{aligned}$$

Then our general product formula will take the form

$$\prod_{i=1}^{n}(d_i - j(i-1)) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})x(x-j)\cdots(x-(k-1)j). \tag{2.17}$$

Because $\{(x)\downarrow_{j,n}\}_{n\geq 0}$ is a basis for the polynomial ring $\mathbb{Q}[x]$, it immediately follows from (2.16) and (2.17) that $\tilde{r}_{n-k,D}^{j} = r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$ for all $0 \leq k \leq n$. We shall show that we can give a completely combinatorial proof of this fact, a result which is best explained through some examples.

In the simplest case, when $d_i \geq j(i-1)$ for $i = 1, \ldots, n$, then it will be the case that the boards D and $\mathcal{B}^{\mathcal{A}}$ are identical. For example, if $j = 2$ and $D = (1,3,6,8)$, then $B = (1,1,2,2)$ and the board $\mathcal{B}^{\mathcal{A}}$ is just the Ferrers board $F(1,3,6,8)$, as in Figure 2.5. In this case, it is easy to see that weight of any placement $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = 1$ so that $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) = |\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})|$. Thus to prove that $\tilde{r}_{n-k,D}^{j} = r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$, we need only find a bijection between $\mathcal{N}_k^{j}(D)$ and $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$. It is easy to see that in the j-attacking Remmel-Wachs model, each rook r in a placement $\mathbb{P} \in \mathcal{N}_k^{j}(D)$ cancels exactly j cells in the each column to its right. Similarly since $a_1 = 0$ and $a_i = j$ for $i \geq 2$, it easy to see that each rook placement $Q \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ also cancels j cells in each column to its right. Thus the only real difference between the two types of rook placements in this case is the exact cells which get cancelled. This suggests a very simple bijection $\Theta^{(j)} : \mathcal{N}_k^{j}(D) \rightarrow \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$. Namely, if $\mathbb{P} \in \mathcal{N}_k^{j}(D)$ with rooks $r_1, \ldots, r_k$ in columns $1 \leq i_1 < \cdots < i_k \leq n$ respectively, then $\Theta^{(j)}(\mathbb{P})$ should be a placements of

Figure 2.6: An example of $\Theta^{(2)}$ in the case where $d_i \geq 2(i-1)$ all i.

rooks $\tilde{r}_1, \ldots, \tilde{r}_k$ in columns $1 \leq i_1 < \cdots < i_k \leq n$ such that for all u, if r_u is in the s_u-th cell in column i_u which is not cancelled by a rook to the left of r_u, reading from bottom to top, then $\tilde{r}_u$ is in the s_u-th cell in column i_u which is not cancelled by a rook to the left of $\tilde{r}_u$, reading from bottom to top. For example, if $\mathbb{P} \in \mathcal{N}_k^2(F(1,3,6,8))$ pictured on the left in Figure 2.6, then its image $\Theta^{(2)}(\mathbb{P})$ is pictured on the right in Figure 2.6. Here is easy to see that left most rook of $\mathbb{P}$ in row 2 of column 2 so that left most rook of $\Theta^{(2)}(\mathbb{P})$ in row two of column 2. We then put a "•" in those cells cancelled by the left most rook in each case. Then we see that rook in column 3 of $\mathbb{P}$ is in the third available cell reading from bottom to top that is available so that the rook in column 3 of $\Theta^{(2)}(\mathbb{P})$ is in the third available cell. We then put an "$*$" in those cells cancelled by the rook in column 3 in each case. Finally, the rook in column 4 in $\mathbb{P}$ is in the second available cell reading from top to bottom so that the rook in column 4 in $\Theta^{(2)}(\mathbb{P})$ is in the second available cell in that column.

In the general case, it may not be the case that $d_i \geq j(i-1)$. If $d_i < j(i-1)$, then $d_i - j(i-1)$ is negative and hence $sgn(i)$ must be negative. It follows that the rooks that are not in the augmented part of the board $\mathcal{B}^{\mathcal{A}}$ in the i-th column will contribute a factor of -1 to the weight of a placement. We will call such columns in $\mathcal{B}$ the *negative columns* of $\mathcal{B}$. If the i-th column of $\mathcal{B}$ is a negative column so that $d_i - j(i-1) < 0$, then clearly $|d_i - j(i-1)| \leq j(i-1)$, then we will call the first $|d_i - j(i-1)|$ cells in the augmented part of $\mathcal{B}^{\mathcal{A}}$ in column i the mirror image of the column i in $\mathcal{B}$. For example, suppose that $j = 2$ and D is the 2-attacking board $F(0,0,1,3,6,7)$. Hence, the product formula for this 2-attacking board is

$$x(x-2)(x+1-4)(x+3-6)(x+6-8)(x+7-10) = \sum_{k=0}^{6} \tilde{r}_{6-k}^{(2)}(x)\downarrow_{2,k} \quad (2.18)$$

Figure 2.7: An example with negative columns and their mirror images.

Thus, the corresponding product formula for the board $\mathcal{B}^{\mathcal{A}}$ will be produced by defining $\mathcal{B} = (0, 2, 3, 3, 2, 3)$, $\mathcal{A} = (0, 2, 2, 2, 2, 2)$, $\overline{sgn}(i) = 1$ for $i = 1, \ldots, 6$ and $sgn(1) = 1$ and $sgn(i) = -1$ for $i > 1$. Then in Figure 2.7, we have pictured the board D and $\mathcal{B}^{\mathcal{A}}$ and have shaded the squares in mirror images of negative columns. In this case, the negative columns are columns 2 through 6. Note, if column i is negative so that $d_i < j(i-1)$, then the first $j(i-1) - d_i$ squares of the augmented part of the board $\mathcal{B}^{\mathcal{A}}$ will be in the mirror image of column i and hence the number of squares in the augmented part of column i which are not in the mirror image is $j(i-1) - (j(i-1) - d_i) = d_i$. Of course, if column i is not negative, then the total number of squares in column i in $\mathcal{B}^{\mathcal{A}}$ is $d_i - j(i-1) + j(i-1) = d_i$. Thus $(d_1, \ldots, d_n)$ represents the column heights, reading from left to right, of either (i) all cells of $\mathcal{B}^{\mathcal{A}}$ in positive column or (ii) all cells in the augmented part of a column that do not lie in the mirror image of negative column.

In the case where there are negative columns, we can define a simple sign-reversing involution I on $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which reduces ourselves to considering only the class of placements $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ in which no rook lies in either in a negative column or the mirror image of the negative column. That is, suppose $\mathbb{P}$ is a placement which contains a rook in a negative column or it mirror image, let r be the left most rook of $\mathbb{P}$ with this property. If r is in the s-th row of the negative part of the column, we let $I(\mathbb{P})$ denote the placement which results in moving r to the s-th row of its mirror image and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = -1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},I(\mathbb{P})}(r) = 1$ so that $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(I(\mathbb{P}))$. If r is in the s-th row of the mirror image of negative part of the column, we let $I(\mathbb{P})$ denote the placement which results in moving r to the s-th row of the negative part of the column and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = 1$ and

Figure 2.8: An example of the involution I.

$w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},I(\mathbb{P})}(r) = -1$ so that once again $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(I(\mathbb{P}))$. Finally, if $\mathbb{P}$ does not have any rooks in either a negative column or its mirror image, then we let $I(\mathbb{P}) = \mathbb{P}$. An example of the involution I, when D is the 2-attacking board $F(0,0,1,3,6,7)$, $\mathcal{B} = (0,2,3,3,2,3)$, $\mathcal{A} = (0,2,2,2,2,2)$, $\overline{sgn}(i) = 1$ for $i = 1,\ldots,6$ and $sgn(1) = 1$ and $sgn(i) = -1$ for $i > 1$, is given in Figure 2.8.

Clearly, $I(I(\mathbb{P})) = \mathbb{P}$, and so

$$\begin{aligned} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) &= \sum_{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \\ &= \sum_{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}),I(\mathbb{P})=\mathbb{P}} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \end{aligned}$$

Since the weight of any $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ such that $I(\mathbb{P}) = \mathbb{P}$ is 1, then we need only show that there is a bijection $\Theta^{(j)}$ from $\mathcal{N}_k^{j}(D)$ to the fixed points of I. But we have already shown that the fixed points of I lies in a region of $\mathcal{B}^{\mathcal{A}}$ whose column heights are $(d_1,\ldots,d_n)$, reading from left to right. Since in both the Remmel-Wachs model and our model, each rook cancel j cells in each column to the right, we can use the same bijection $\Theta^{(j)}$ described above to give the desired bijection between these two sets of rook placements.

Goldman-Haglund j-Creation Boards

Next we consider the j-creation model of Goldman and Haglund [10]. If we fix j and start with a Ferrers board $D = (d_1,\ldots,d_n)$, then the product formula that arises out of the j-creation model in this case is

$$\prod_{i=1}^{n}(x + d_i + (j-1)(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(j)}(D)(x)\uparrow_{k,j} \tag{2.19}$$

Figure 2.9: An example of the difference of shapes between the j-creation board $B^{(3)}$ with $B = F(0,1,2,3,3)$ and the corresponding augmented rook board.

In this case, to get an equivalent product formula in our model, we must let $\mathcal{B} = (d_1, d_2{+}j, d_3{+}2j, \ldots, d_n{+}j(n{-}1))$, $\mathcal{A} = (0, j, j, \ldots, j)$ and $sgn(i) = \overline{sgn}(i) = 1$ for all i so that our product formula will become

$$\prod_{i=1}^{n}(x + d_i + j(i-1)) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})(x)\uparrow_{k,j}. \qquad (2.20)$$

Since $\{(x)\uparrow_{n,j}\}_{n\geq 0}$ is a basis for the polynomial ring $\mathbb{Q}[x]$, it immediately follows from (2.19) and (2.20) that $r^{(j)}_{n-k}(D) = r^{\mathcal{A}}_{n-k}(sgn, \overline{sgn}, \mathcal{B}^{\mathcal{A}})$.

Again our goal is the give a completely combinatorial proof of this fact. To do this, we will follow the same steps as the we did for the general case of the Remmel-Wachs model, namely, will first define an involution J on $\mathcal{N}^{\mathcal{A}}_k(\mathcal{B}^{\mathcal{A}})$ and then give a bijection between $\mathcal{N}^{(j)}(D)$ and the fixed points of J. As before, these steps are best illustrated through an example. We will let $j = 3$ and consider the 3-creation board $B^{(3)}$ with $B = F(0,1,2,3,3)$. Thus the product formula for $B^{(3)}_x$ in this case is

$$x(x+3)(x+6)(x+9)(x+11) = \sum_{k=0}^{5} r^{(3)}_{5-k}(B)x\uparrow_{(k,2)}. \qquad (2.21)$$

From our standpoint of augmented rook boards, to generate the same product formula we would set $\mathcal{B} = (0,3,6,9,11)$, $\mathcal{A} = (0,2,2,2,2)$, and $sgn(i) = \overline{sgn}(i) = +1$ for every j. If we then construct $\mathcal{B}^{\mathcal{A}}$, we can see the vast difference in shape between these two boards, pictured in Figure 2.9.

Note also that our weighting ensures that each rook in the augmented part of the board has weight -1 and each rook in $\mathcal{B}$-part of the board has weight 1. For each rook placement $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_k(\mathcal{B}^{\mathcal{A}})$, we will define the mirror image of the augmented part of $\mathcal{B}^{\mathcal{A}}$ in the i-th column relative to $\mathbb{P}$ to be the consists the top s cells in the $\mathcal{B}$-part of column i if there are s cells in the augmented part of the column i which are not cancelled by the rooks to the left of column i. In

Figure 2.10: An example of the involution J.

Figure 2.10, we have pictured a placement $\mathbb{P}$ which has rooks in columns 2 and 4. We have placed a "•" in the cells cancelled by the rook in column 2 and we have placed an "$*$" in the cells cancelled by the rooks in column 4. We have also shaded, in the $\mathcal{B}$-part of the board, each cell which is in the mirror image of the uncancelled cells in the augmented part of its column. Then the involution J is very simple. That is, suppose $\mathbb{P}$ is a placement which contains a rook in an cell in the augmented part of $\mathcal{B}^{\mathcal{A}}$ or it mirror image. Let r be the right most rook of $\mathbb{P}$ with this property. If r is in the s-th row, reading from bottom to top, in the augmented part of the column, we let $J(\mathbb{P})$ denote the placement which results in moving r to the s-th row of its mirror image, reading from bottom to top, and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = -1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},J(\mathbb{P})}(r) = 1$ so that $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(J(\mathbb{P}))$. If r is in the s-th row, reading bottom to top, of the mirror image of the uncancelled cells in the augmented part of its column, we let $J(\mathbb{P})$ denote the placement which results in moving r to the s-th row of the uncancelled cells in the augmented part of its column, reading bottom to top, and leaving all other rooks in the same place. Note that in this case, $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},\mathbb{P}}(r) = 1$ and $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}},J(\mathbb{P})}(r) = -1$ so that once again $w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(J(\mathbb{P}))$. Finally, if $\mathbb{P}$ does not have any rooks in either an uncancelled cell in the augmented part or its mirror image in any column, then we let $J(\mathbb{P}) = \mathbb{P}$. In Figure 2.10, we have pictured $J(\mathbb{P})$ on the right.

Clearly, $J(J(\mathbb{P})) = \mathbb{P}$, and thus,

$$\begin{aligned} r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}) &= \sum_{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \\ &= \sum_{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}),J(\mathbb{P})=\mathbb{P}} w_{sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \end{aligned}$$

Since the weight of any $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ such that $J(\mathbb{P}) = \mathbb{P}$ is 1, we need only show that there is a bijection Δ from the fixed points of J to $\mathcal{N}_k^{(j)}(D)$.

In this case, the bijection Δ can be constructed by recursion. That is, suppose that $D = F(d_1, \ldots, d_n,)$ is a Ferrers board where $n \geq 2$. Then we claim that for $0 \leq k \leq n$, we have the following recursion among rook numbers in the j-creation model of Goldman and Haglund [10].

$$r_k^{(j)}(F(d_1, \ldots, d_n)) = r_k^{(j)}(F(d_1, \ldots, d_{n-1})) + (d_n + j(k-1))r_{k-1}^{(j)}(F(d_1, \ldots, d_{n-1})). \tag{2.22}$$

The recursion given in (2.22) is the result of classifying the rook placements in $\mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ according to whether there is a rook in the last column or not. If $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ is a rook placement with no rooks in the last column, then we can also regard $\mathbb{P}$ as a placement in $\mathcal{N}_k^{(j)}(F(d_1, \ldots, d_{n-1}))$. However if $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ is a rook placement with has a rook in the last column, then we let $Q \in \mathcal{N}_{k-1}^{(j)}(F(d_1, \ldots, d_{n-1}))$ be the rook placement which consists of the rooks of $\mathbb{P}$ in the first $n-1$ columns. For each such Q, there will be a total of $d_n + j(k-1)$ rows in which to place a rook in the last column since each of the $k-1$ rooks in Q create j new rows in which to place a rook.

Next consider the two sequences of length n, $\mathcal{B}_n = (d_1, d_2+j, \ldots, d_n+j(n-1))$ and $\mathcal{A}_n = (0, j, \ldots, j)$ versus the two sequences of length $n-1$, $\mathcal{B}_{n-1} = (d_1, d_2 + j, \ldots, d_{n-1} + j(n-2)$ and $\mathcal{A}_{n-1} = (0, j, \ldots, j)$. We claim that if

$$\tilde{r}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) = |\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}|, \tag{2.23}$$

then,

$$\tilde{r}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) = \tilde{r}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}) + (d_n + j(k-1))\tilde{r}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}). \tag{2.24}$$

Again, this recursion is the result of classifying the rook placements in $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ according to whether or not there is a rook in the last

Figure 2.11: The recursive deconstruction of $\mathbb{P} \in \mathcal{N}_3^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$.

column. That is, if $R \in \{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ has no rook in the last column, then R can be viewed as an element of $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}}) : J(\mathbb{P}) = \mathbb{P}\}$. On the other hand, if $\mathbb{P}$ does have a rook in its last column, then let Q be the placement that is the restriction of $\mathbb{P}$ to the first $n-1$ columns. It is easy to check that our definition of the involution J for $\mathcal{N}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ ensures that $Q \in \mathcal{N}_{k-1}^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ and $J(Q) = Q$. Moreover, we claim that number of ways to extend Q to placement in $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ is $d_n + j(k-1)$. Note that the n-th column of $\mathcal{B}_n^{\mathcal{A}_n}$ has height $d_n + j(n-1) + j(n-1)$ with the top $j(n-1)$ cells being in the augmented part of the board. Each of the rooks in Q cancels j cells in the augmented part of $\mathcal{B}_n^{\mathcal{A}_n}$ in the n-th column. Thus there are $j(n-1) - j(k-1) = j(n-k)$ cells in the augmented part of $\mathcal{B}_n^{\mathcal{A}_n}$ in the n-th column which are not cancelled so that the mirror image of these cells is the top $j(n-k)$ cells in the $\mathcal{B}_n$-part of $\mathcal{B}_n^{\mathcal{A}_n}$. Since for a fixed point of J, we are not allowed to place a rook in either the augmented part of the n-th column or the mirror image of its uncancelled cells, it follows that we have a total of $d_n + j(n-1) - j(n-k) = d_n + j(k-1)$ ways to extend Q to a placement $\mathbb{P} \in \{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$.

Our proofs of the recursions (2.22) and (2.24) easily lead to a recursive way to define our desired bijection Δ from $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ to $\mathcal{N}_k^{(j)}(D)$. That is, we start with a $\mathbb{P} \in \{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}_n}(\mathcal{B}_n^{\mathcal{A}_n}) : J(\mathbb{P}) = \mathbb{P}\}$ and consider the sequence j-creation rook placements $\mathbb{P} = \mathbb{P}_n, \mathbb{P}_{n-1}, \ldots \mathbb{P}_1$ where $\mathbb{P}_i$ is just the restriction of $\mathbb{P}$ to the first i columns for $i = 1, \ldots, n$. For example, we go back to the case where $j = 3$, the 3-creation board $B^{(3)}$ is $D = F(0,1,3,3,3)$, $\mathcal{B} = (0,3,6,9,11)$, $\mathcal{A} = (0,2,2,2,2)$, and $sgn(i) = \overline{sgn}(i) = +1$ for every j. Then the sequence $\mathbb{P} = \mathbb{P}_5, \mathbb{P}_4, \mathbb{P}_3, \mathbb{P}_2, \mathbb{P}_1$ is pictured in Figure 2.11.

Now, the image $\Delta(\mathbb{P})$ will be obtained by constructing an element of $\mathcal{N}_k^{(j)}(D)$. By performing the analogous steps to build up $\Delta(\mathbb{P})_1, \ldots, \Delta(\mathbb{P})_n = \Delta(\mathbb{P})$. That

Figure 2.12: The recursive construction of $\Delta(\mathbb{P}) \in \mathcal{N}_3^{(j)}(D)$.

is, at step 1, if $\mathbb{P}_1$ has no rook in the first column, then $\Delta(\mathbb{P})_1$ has no rook in the first column. Otherwise, the length of the first column in both the restriction of D to the first column and $\mathcal{B}^{\mathcal{A}}$ is the same, namely d_1, so that if the rook in $\mathbb{P}_1$ is in the r-th row, reading from top to bottom, then we place the rook in $\Delta(\mathbb{P})_1$ in the r-th row, reading from top to bottom. In general, having constructed $\Delta(\mathbb{P})_1, \ldots, \Delta(\mathbb{P})_i$, then if $\mathbb{P}_{i+1}$ has no rook in the column $i+1$, then $\Delta(\mathbb{P})_{i+1}$ also has no rook in column $i+1$ and the restriction of $\Delta(\mathbb{P})_{i+1}$ to the first i columns is just $\Delta(\mathbb{P}_i)$. Otherwise, if the rook in the $(i+1)$-st column of $\mathbb{P}_{i+1}$ is in the r-th available square reading from bottom to top, then we set $\Delta(\mathbb{P})_{i+1}$ to be the rook placement which results by extending $\Delta(\mathbb{P})_i$ by adding a rook in the column $i+1$ which is in the r-th available cell, reading from top to bottom. An example of the sequence $\Delta(\mathbb{P})_1, \ldots, \Delta(\mathbb{P})_5$ for the $\mathbb{P}$ pictured in Figure 2.11 is given in Figure 2.12.

Thus, in both the j-cancellation model of Remmel and Wachs and j-creation model of Goldman and Haglund, one can give direct combinatorial proofs of the fact that the rooks numbers that appear in the product formulas for those models are the same that as the rook numbers that appear in the corresponding product formulas in our model. However, we should point out that our model gives rise to a much wider class of product formulas than can be derived in either of those two models. That is, in the j-cancellation model of Remmel-Wachs, the product formulas holds only for j-attacking Ferrers boards. Similarly, in the j-creation model of Goldman and Haglund, the product formula holds only for Ferrers boards. However, there is no such restrictions on the boards in our model that give rise to product formulas.

Chapter 3

Q-Analogues of the General Product Formula

3.1 Introduction

In this chapter, we shall describe how we can derive several q-analogues of the general product formula described in Chapter 2 by q-counting rook placements considered in that chapter. That is, for any $n \in \mathbb{N}$, we again define $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and we use the convention that $[-n]_q := -[n]_q$. We also set $\overline{A}_i := \sum_{s=1}^{i} \overline{sgn}(s) a_s$. Our first goal is to show that for an appropriate q-analogue of the rook number $r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$, $r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q)$, we can prove the following q-analogue of Theorem 2.1.

Theorem 3.1. *Suppose $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are two sequences of nonnegative integers, and let $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$ be two sign functions. Then,*

$$\prod_{i=1}^{n}([x]_q + sgn(i)[b_i]_q) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{s=1}^{k}([x]_q + [\overline{A}_s]_q). \tag{3.1}$$

We will refer to Equation (3.1) as the *q-general product formula*. Then we shall relate the q-general product formula to other q-analogues of product formulas that have appeared in the literature.

3.2 A General Q-Analogue Product Formula

3.2.1 The Q-Weighting Of General Rook Placements $\&$ An Example

Fix the two sequences $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ and the two sign functions $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$. The first step in proving Equation 3.1 is to define a q-weight, $\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$, of each placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ and a q-weight, $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q)$, to each placement $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$. To do this, we shall define a q-weight, $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$, to each cell c in $\mathcal{B}_x^{\mathcal{A}}$. Then if $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_k$, we set

$$\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{k} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{3.2}$$

Similarly, if $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_n$, then

$$\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q) = \prod_{i=1}^{n} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{3.3}$$

Then we define

$$r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{3.4}$$

To define $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$, we proceed as follows:

1. For each i, the weights $\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(c)$ of the cells in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $1, q, q^2, \ldots, q^{x-1}$, reading from bottom to top.

2. For each i, the weights $\mu_{q,\mathcal{B}_x^A}(c)$ of the cells in the i-th column of the $\mathcal{B}$-part of $\mathcal{B}_x^A$ are $sgn(i)$, $sgn(i)q$, $sqn(i)q^2, \ldots, sgn(i)q^{b_i-1}$, reading from bottom to top.

3. For each i, we assign weight $\mu_{q,\mathcal{B}_x^A}(c)$ to cells in the i-th column of the lower augmented part as follows. First, we assign the weight $\overline{sgn}(i)1$, $\overline{sgn}(i)q$, $\overline{sgn}(i)q^2$, …, $\overline{sgn}(i)q^{a_1-1}$ to the cells in a_1-st part of column i in the lower augmented board reading from top to bottom. Thus the sum of the q-weights of cells in a_1-st part of column i in the lower augmented board is $[\overline{sgn}(i)a_1]_q$. Next suppose that we have assigned the weights to cells in a_j-th part of column i in the lower augmented part for $j = 1, \ldots, s$ so that the sum of the q-weights of cells that lie in a_j-th part of column i in the lower augmented board for $j \leq s$ is $[\overline{A}_s]_q$. Then we defined the weights to the cells in a_{s+1}-th part of column i in the lower augmented part according to the following cases:

   Case 1. $0 \leq \overline{A}_s \leq \overline{A}_{s+1}$. In this case, we assign the weight of cells in the a_{s+1} part to be $q^{\overline{A}_s}, q^{\overline{A}_s+1}, \ldots, q^{\overline{A}_{s+1}-1}$, reading from top to bottom, .

   Case 2. $0 \leq \overline{A}_{s+1} < \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $-q^{\overline{A}_s-1}, -q^{\overline{A}_s-2}, \ldots, -q^{\overline{A}_{s+1}}$, reading from top to bottom.

   Case 3. $\overline{A}_{s+1} < 0 \leq \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $-q^{\overline{A}_s-1}, -q^{\overline{A}_s-2}, \ldots, -q^0, -q^1, \ldots - q^{|\overline{A}_{s+1}|}$, reading from top to bottom.

   Case 4. $0 \geq \overline{A}_s \geq \overline{A}_{s+1}$. In this case, we assign the weight of cells in the a_{s+1} part to be $-q^{\overline{A}_s}, -q^{\overline{A}_s+1}, \ldots, -q^{\overline{A}_{s+1}-1}$, reading from top to bottom, .

   Case 5. $0 \geq \overline{A}_{s+1} > \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $q^{\overline{A}_s-1}, q^{\overline{A}_s-2}, \ldots, q^{\overline{A}_{s+1}}$, reading from top to bottom.

Figure 3.1: The q-weighting of cells in placements in $\mathcal{B}_x^{\mathcal{A}}$

   Case 6. $\overline{A}_{s+1} > 0 \geq \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $q^{\overline{A}_s-1}, q^{\overline{A}_s-2}, \dots, q^0, q^1, \dots q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

4. For each i, the cell in the r-th row of the i-th column of the upper augmented part of the board, reading from bottom to top, is equal to -1 times the weight of the cell in r-row of i-th column of the lower augmented part of the board, reading from top to bottom. That is, in the upper augmented part of column i, the weights of the cells in the i-th column are the exact mirror image as those which come from the lower augmented part.

An example of this kind of labeling can be seen in the left-hand side of Figure 3.2. Here we have used the same board and rook placement as in Figure 2.4, where $x = 4$, $\mathcal{B} = (1, 2, 2, 4)$, $\mathcal{A} = (2, 1, 2, 1)$,

$$sgn(i) = \begin{cases} +1 & \text{if } i = 1, 2, 4, \\ -1 & \text{if } i = 3 \end{cases}$$

and

$$\overline{sgn}(i) = \begin{cases} +1 & \text{if } i = 2, \\ -1 & \text{if } i = 1, 3, 4. \end{cases}$$

For the q-weighting of these placements, if we consider the placement, $\mathbb{P}$, in the right-hand side of Figure 3.2 , then we can see that the rooks placed from left to right correspond to the cell labellings $1, q^2, q, -1$ respectively, and thus this placement has weight

$$\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = (1)(q^2)(q)(-1) = q^3.$$

The key property of our weighting of cells in $\mathcal{B}_x^{\mathcal{A}}$, which is easily established by induction, is that the sum of the q-weights of cells that lie in a_j-th part of column i in the lower augmented board for $j \leq s$ is $[\overline{A}_s]_q$ and the sum of the q-weights of cells that lie in a_j-th part of column i in the upper augmented board for $j \leq s$ is $[-\overline{A}_s]_q$.

3.2.2 A Proof of Equation (3.1)

Proof of Equation (3.1): We would like to compute the polynomial

$$G(q) := \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \tag{3.5}$$

in two different ways.

If we first place the rooks starting with the leftmost column and working to the right, then we can see that in the first column, the sum of the q-weights of the cells in the first column are $[x]_q + [sgn(1)b_1]_q + [\overline{sgn}(1)a_1]_q + [-\overline{sgn}(1)a_1]_q = [x]_q + [sgnb_1]_q$. When we go to place a rook in the second column, we have two cases.

Case I: Suppose the rook that was placed in the first column was placed below the high bar. Then nothing was cancelled in the second column, so we can place a rook in any cell of the second column. It follows that our weighting ensures that the sum of the q-weights of the cells in the second column are $[x]_q + [sgn(2)b_2]_q + [\overline{A}_2]_q + [-\overline{A}_2]_q = [x]_q + [sgn(2)b_2]_q$.

Case II: If the rook placed in the first column was placed above the high bar, then the cells corresponding to a_2 in both the upper and lower augmented parts of the second column are cancelled. It follows that our weighting ensures that the sum of the q-weights of the cells in the second column are $[x]_q + [sgn(2)b_2]_q + [\overline{A}_1]_q + [-\overline{A}_1]_q = [x]_q + [sgn(2)b_2]_q$.

In general, suppose we are placing a rook in the j^{th} column, and suppose that we have placed s rooks above the high bar and t rooks below the high bar in the first $j-1$ columns. Then in the j^{th} column we have, by Lemma 2.2, $x + b_j + 2(A_{t+1})$ choices as to where to place the rook in that column. Again, it follows that our weighting ensures that the sum of the q-weights of the cells in the second column are $[x]_q + [sgn(j)b_j]_q + [\overline{A}_{t+1}]_q + [-\overline{A}_{t+1}]_q = [x]_q + [sgn(j)b_j]_q$.

Thus, it follows that

$$G(q) = \prod_{i=1}^{n}([x]_q + [sgn(i)b_i]_q)$$

which gives the left-hand side of (3.1).

The second way of counting over the weights of all the rook placements in $\mathcal{B}_x^{\mathcal{A}}$ is to organize the placements by how many rooks lie above the high bar. Suppose that we fix a placement $\mathbb{P}$ of $n-k$ non-attaching rooks in $\mathcal{B}^{\mathcal{A}}$. Then we wish to compute

$$\sum_{\substack{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q). \tag{3.6}$$

Clearly each such Q in the sum arises from $\mathbb{P}$ by placing rooks below the high bar in the remaining columns. Thus, there are k columns left that need to have rooks placed in them, below the high bar. We will place the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 2.3, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_k)$. However, it follows by the properties of our assignment of q-weights to the cells that

$$\sum_{\substack{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}) \\ Q \cap \mathcal{B}^{\mathcal{A}} = \mathbb{P}}} \mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q) = ([x]_q + [\overline{A}_1]_q)([x]_q + [\overline{A}_2]_q)\cdots([x]_q + [\overline{A}_k]_q). \tag{3.7}$$

Moreover, when any such placement Q is extended to a placement $\mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$, by combining it with $\mathbb{P}$, the resulting q-weight will simply be $\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})\mu_{q,\mathcal{B}_x^{\mathcal{A}}}(Q)$. Thus,

$$\begin{aligned} G(q) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})([x]_q + [\overline{A}_1]_q)([x]_q + [\overline{A}_2]_q) \cdots ([x]_q + [\overline{A}_k]_q) \\ &= \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q)([x]_q + [\overline{A}_1]_q)([x]_q + [\overline{A}_2]_q) \cdots ([x]_q + [\overline{A}_k]_q) \end{aligned}$$

which gives the right-hand side of (3.1). □

3.3 Modified Q-Analogues of the Product Formula

We note that in our q-general product formula, (3.1), we have taken the q-analogue of $(x+a)$ to be $[x]_q + [a]_q$ and the q-analogue of $(x-a)$ to be $[x]_q - [a]_q$ if x and a are non-negative integers. However, if one looks at the Garsia and Remmel's q-analogue of the product formula one takes the q-analogue of $x-a$ to be $[x-a]_q$ and in the q-analogue of Goldman and Haglund's j-creation model, one takes the q-analogue of $(x+a)$ to $[x+a]_q$. It turns out the we can easily modify our q-analogue of the q-general product formula (3.1) to produce q-analogues of our general product formula formulas where we take the q-analogue of $(x+a)$ to be $[x+a]_q$ and the q-analogue of $(x-a)$ to be $[x-a]_q$ by some simple transformations of our formulas and q-rook numbers.

The basic idea to transform our q-analogue of the general product formula (3.1) is based on the following simple identities which hold when x and a are non-negative integers with $x \geq a$:

$$[x]_q - [a]_q = q^a [x-a]_q \tag{3.8}$$

and

$$[x]_q + q^x [a]_q = [x+a]_q. \tag{3.9}$$

We shall first explore how these transformations force us to modify our q-rook numbers to prove q-analogues of our general product formula formulas where we take the q-analogue of $(x+a)$ to be $[x+a]_q$ and the q-analogue of $(x-a)$ to be $[x-a]_q$. Throughout this section, we shall fix two sequences of nonnegative integers $\mathcal{B}=(b_1,\ldots,b_n)$ and $\mathcal{A}=(a_1,\ldots,a_n)$.

3.3.1 Case I: $sgn(i)=\overline{sgn}(i)=-1$

In the case where $sgn(i)=\overline{sgn}(i)=-1$ for every $1\leq i\leq n$, we see that $\overline{A}_i=\sum_{j=1}^{i}\overline{sgn}(i)a_i=-A_i$ where $A_i=a_1+\cdots a_i$. Thus (3.1) becomes

$$\prod_{i=1}^{n}([x]_q-[b_i]_q)=\sum_{k=0}^{n}r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn},q)\prod_{s=1}^{k}([x]_q-[A_s]_q). \tag{3.10}$$

Now if $x\geq max(\{b_i,A_i:i=1,\ldots,n\})$, then (3.10) becomes

$$\prod_{i=1}^{n}q^{b_i}[x-b_i]_q=\sum_{k=0}^{n}r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn},q)\prod_{s=1}^{k}(q^{A_s}[x-A_s]_q). \tag{3.11}$$

If we now replace $r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn})$ with $\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q)$ where

$$\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q):=q^{(A_1+A_2+\cdots+A_{n-k})-(b_1+\cdots+b_n)}r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},sgn,\overline{sgn}),$$

we obtain the following q-analogue of Equation (2.3):

$$\prod_{i=1}^{n}([x-b_i]_q)=\sum_{k=0}^{n}\hat{r}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}},q)([x-A_1]_q)([x-A_2]_q)\cdots([x-A_k]_q). \tag{3.12}$$

We note that in this case, our weighting of cells in $\mathcal{B}_x^{\mathcal{A}}$ is very simple.

1. For each i, the weights of the cells in i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ is $1,q,\ldots,q^{x-1}$ reading from bottom to top.

2. For each i, the weights of the cells in i-th column of the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$ is $-1,-q,\ldots,-q^{b_i-1}$ reading from bottom to top.

3. For each i, the weights of the cells in i-th column of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ is $-1,-q,\ldots,-q^{A_i-1}$ reading from top to bottom.

Figure 3.2: The q-weighting of cells in placements in $\mathcal{B}_x^{\mathcal{A}}$ when $sgn(i) = \overline{sgn}(i) = -1$.

4. For each i, the weights of the cells in i-th column of the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ is $1, q, \ldots, q^{A_i - 1}$ reading from bottom to top.

These weights are pictured in Figure 3.2 in the case where $\mathcal{B} = (0, 1, 3, 3)$ and $\mathcal{A} = (1, 2, 1, 2)$.

We note that in the j-attacking case of the Remmel-Wachs model one can get similar formulas in the case where we start with a j-attacking Ferrers board $D = F(d_1, \ldots, d_n)$ where $d_i \leq j(i-1)$ for all i. In particular, it must be that $d_1 = 0$. In such a case, the q-analogue of the product formula, which is just the $p = 1$ case of the (1.83), is

$$[x]_q[x - (j - d_2)]_q \cdots [x - ((n-1)j - d_n)]_q = \sum_{k=0}^{n} \tilde{r}^{j}_{k,B}(1, q)[x]_q \downarrow_{n-k,j} \tag{3.13}$$

To get the equivalent product formula out of (3.11), we must set $\mathcal{B} = (0, j - d_2, \ldots, j(n-1) - d_n)$ and $\mathcal{A} = (0, j, \ldots, j)$ with $sgn(i) = \overline{sgn}(i) = -1$ for all i. In that case, (3.11) becomes

$$[x]_q[x - (j - d_2)]_q \cdots [x - ((n-1)j - d_n)]_q = \sum_{k=0}^{n} \hat{r}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}}, q)[x]_q \downarrow_{n-k,j} \tag{3.14}$$

If one thinks of $[x - a]$ as $\frac{q^x q^{-a} - 1}{q - 1}$, then one can think of (3.11) and (3.14) as formulas involving the variable q^x with coefficients which are in the field $\mathbb{Q}(q)$. That is, we can think of (3.11) and (3.14) as identities in $\mathbb{Q}(q)[q^x]$. Thus, since $\{[x_q]\downarrow_n\}_{n \geq 0}$ is basis for $\mathbb{Q}(q)[q^x]$, it follows that $\hat{r}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}}, q) = \tilde{r}^{j}_{k,B}(1, q)$ for all k in this case. In fact, we can give a completely combinatorial proof of this fact.

Recall the involution I that we used to show that the rook numbers our model and the j-attacking Remmel-Wacks model in the $q = 1$ case were equivalent. In the current case, we are assuming that every non-empty column is negative. Moreover, we see that our q-weighting of cells in this case ensures that the weight of the cell in in r-th row in a negative column is $-q^{r-1}$ while its mirror image in the augmented part, i.e., the r-th cell in augmented part of that column, reading from bottom to top, has weight q^{r-1}. It follows that our sign reversing involution I preserves the q-weight so that if $I(\mathbb{P}) \neq \mathbb{P}$ for some $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$, then $\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = -\mu_{q,\mathcal{B}^{\mathcal{A}}}(I(\mathbb{P}))$. Thus I shows that

$$\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) = \sum_{\substack{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) \\ I(\mathbb{P}) = \mathbb{P}}} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{3.15}$$

Moreover, if $I(\mathbb{P}) = \mathbb{P}$, then all the rooks $\mathbb{P}$ must lie in the augmented part of $\mathcal{B}^{\mathcal{A}}$. Since the weights of the cells in augmented part of $\mathcal{B}^{\mathcal{A}}$ are just powers of q with no negative signs, it follows that in order to prove that $\hat{r}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = \tilde{r}_{k,B}^{j}(1, q)$, we need only construct a weight preserving bijection from $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) : I(\mathbb{P}) = \mathbb{P}\}$ onto $\mathcal{N}^{(j)}(D)$. We shall see that much like we did in the case of j-creation model in Section 2.4.1, we can construct such a map by recursion.

First suppose that $D_n = F(d_1, \ldots, d_n)$ is a j-attacking Ferrers board. Then we have that

$$\begin{aligned} \tilde{r}_{k,F(d_1,\ldots,d_n)}^{j}(1,q) &= q^{d_n - jk} \tilde{r}_{k,F(d_1,\ldots,d_{n-1})}^{j}(1,q) \\ &\quad + [d_n - j(k-1)]_q \tilde{r}_{k-1,F(d_1,\ldots,d_{n-1})}^{j}(1,q). \end{aligned} \tag{3.16}$$

This recursion is proved by partitioning the placements $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1, \ldots, d_n))$ based on whether $\mathbb{P}$ has a rook in column n. That is, if $\mathbb{P}$ does not have a rook in column n, then $\mathbb{P}$ can be regarded as a placement in $\mathcal{N}_k^{(j)}(F(d_1, \ldots, d_{n-1}))$. Then, from the definition of the weight function $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}$ we see that

$$\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P}) = q^{d_n - jk} \tilde{W}_{1,q,F(d_1,\ldots,d_{n-1})}(\mathbb{P}), \tag{3.17}$$

since there are a total of $d_n - jk$ uncancelled cells in the last column and each to these cells contributes a factor of q to $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P})$. Thus the sum of

$\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P})$ over all placements $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1,\ldots,d_n))$ which do not have a rook in last column is $q^{d_n-jk}\tilde{r}^j_{k,F(d_1,\ldots,d_{n-1})}(1,q)$. Now if $\mathbb{P}$ does have a rook in the last column and Q is the restriction of $\mathbb{P}$ to the first $n-1$ columns of D, then Q can be regarded as a placement in $\mathcal{N}_{k-1}^{(j)}(F(d_1,\ldots,d_{n-1}))$. Moreover, if we want to extend Q to a placement $\mathbb{P}^* \in \mathcal{N}_k^{(j)}(F(d_1,\ldots,d_n))$, then we can place the rook in any of the $d_n - j(k-1)$ cells in the last row since there are exactly $d_n - j(k-1)$ cells in column n which are not cancelled by the rooks in Q. The rook in the last column will contribute a factor of $1, q, q^2, \ldots, q^{d_n-j(k-1)-1}$ to weight $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P}^*)$ depending on whether it is placed in highest row with an uncancelled cell, second highest with an uncancelled cell, etc.. It follows that the sum of $\tilde{W}_{1,q,F(d_1,\ldots,d_n)}(\mathbb{P})$ over all placements $\mathbb{P} \in \mathcal{N}_k^{(j)}(F(d_1,\ldots,d_n))$ which do have a rook in last column is $[d_n - j(k-1)]_q\tilde{r}^j_{k-1,F(d_1,\ldots,d_{n-1})}(1,q)$.

Now assume that $d_i \leq j(i-1)$ for $i = 1,\ldots,n$, and let $\mathcal{B} = (d_1, j-d_2, \ldots, j(n-1)-d_n)$, $\mathcal{A} = (0, j, \ldots, j)$, and $sgn(i) = \overline{sgn}(i) = -1$ for all i. Thus $A_1 = 0$ and $A_i = j(i-1)$ for $i \geq 0$. Let

$$\hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = \sum_{\substack{\mathbb{P}\in\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) \\ I(\mathbb{P})=\mathbb{P}}} \mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{3.18}$$

Then we claim that

$$\begin{aligned}\hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) &= q^{d_n-kj}\hat{\hat{r}}_k^{(0,j,\ldots,j)}((d_1, j-d_2, \ldots, j(n-2)-d_{n-1})^{(0,j,\ldots,j)}, q) \\ &\quad +[d_n - j(k-1)]_q\hat{\hat{r}}_{k-1}^{(a_1,\ldots,a_{n-1})}((b_1,\ldots,b_{n-1})^{(a_1,\ldots,a_{n-1})}, q).\end{aligned} \tag{3.19}$$

That is, we can partition the placements $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ into two sets depending on whether there is a rook in the last column or not. If $\mathbb{P}$ does not have a rook in the last column, then $\mathbb{P}$ can be regarded as a placement in $\mathcal{N}_k^{\mathcal{A}_{n-1}}(\mathcal{B}_{n-1}^{\mathcal{A}_{n-1}})$ such that $I(\mathbb{P}) = \mathbb{P}$. Then our definitions ensure that

$$\begin{aligned}
& q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \\
= \;& q^{A_{n-k}-(j(n-1)-d_n)}q^{A_1+\cdots+A_{n-1-k}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,(d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)}}(\mathbb{P}) \\
= \;& q^{j(n-k-1)-(j(n-1)-d_n)} \\
& \times q^{A_1+\cdots+A_{n-1-k}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,(d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)}}(\mathbb{P}) \\
= \;& q^{d_n-kj}q^{A_1+\cdots+A_{n-1-k}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,(d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)}}(\mathbb{P}).
\end{aligned}$$

Hence it follows that the sum of the $q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$ over all placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which have no rook in the last column and $I(\mathbb{P}) = \mathbb{P}$ is $q^{d_n-kj}\hat{\hat{r}}_k^{(0,j,\ldots,j)}((d_1, j-d_2, \ldots, j(n-2)-d_{n-1})^{(0,j,\ldots,j)}, q)$. Now if $\mathbb{P}$ does have a rook in its last column, then let Q be the restriction of $\mathbb{P}$ to first $n-1$ columns. Then Q can be regarded as a placement in $\mathcal{N}_{k-1}^{(a_1,\ldots,a_{n-1})}((b_1,\ldots,b_{n-1})^{(a_1,\ldots,a_{n-1})})$ for which $I(Q) = Q$. We can extend Q to placement $\mathbb{P}^* \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ such that $I(\mathbb{P}^*) = \mathbb{P}^*$ in $d_n - j(k-1)$ ways. That is, there are a total of $j(n-1)$ cells in the augmented part of the last column. Of those cells in the augmented part, the first $j(n-1)-d_n$ are in the mirror image of the $\mathcal{B}$-part of the n-th column so that we have a total of $j(n-1)-(j(n-1)-d_n) = d_n$ cells in the top of the column where we can place a rook. Then each rook in Q will cause the top $j(k-1)$ cells in the augmented part of column n to be cancelled. Thus we can only have $d_n - j(k-1)$ cells the augmented part of column n in which to place a rook for $\mathbb{P}^*$. Those cells are in rows $(j(n-1)-d_n)+1, \ldots, j(n-1)-d_n+d_n-j(k-1)$ which have q-weights $q^{(j(n-1)-d_n)}, \ldots, q^{j(n-1)-j(k-1)-1}$ respectively. Now whenever we place rook in the cell with weight $q^{(j(n-1)-d_n)+i}$ for $i = 0, \ldots, d_n - j(k-1) - 1$ to obtain a rook placement $\mathbb{P}^*$, then we have that

$$\begin{aligned} & q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}^*) \\ = \;& q^{(j(n-1)-d_n)+i}q^{-(j(n-1)-d_n)} \\ & \times q^{A_1+\cdots+A_{n-1-(k-1)}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,(d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)}}(Q) \\ = \;& q^{(j(n-1)-d_n)+i}q^{-(j(n-1)-d_n)} \\ & \times q^{A_1+\cdots+A_{n-1-(k-1)}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,(d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)}}(Q) \\ = \;& q^i q^{A_1+\cdots+A_{n-1-(k-1)}-\sum_{i=1}^{n-1}(j(i-1)-d_i)}\mu_{q,(d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)}}(Q). \end{aligned}$$

It then follows that the sum of the $q^{A_1+\cdots+A_{n-k}-\sum_{i=1}^{n}(j(i-1)-d_i)}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$ over all placements $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ which have a rook in the last column and $I(\mathbb{P}) = \mathbb{P}$ is

$$\begin{aligned} & (1+q+\cdots q^{d_n-j(k-1)-1})\hat{\hat{r}}_{k-1}^{(0,j,\ldots,j)}((d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)},q) \\ & = [d_n-j(k-1)]_q\hat{\hat{r}}_{k-1}^{(0,j,\ldots,j)}((d_1,j-d_2,\ldots,j(n-2)-d_{n-1})^{(0,j,\ldots,j)},q). \end{aligned}$$

Our proofs of the recursion (3.16) and (3.20) show that we can recursively construct a weight bijection Θ from $\mathcal{N}_k^{(j)}(D)$ onto $\{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}) : I(\mathbb{P}) = \mathbb{P}\}$. That is, given a placement $Q \in \mathcal{N}_k^{(j)}(D)$, let $Q = Q_n, \ldots, Q_1$ be the sequence of placements that results by letting Q_i be the restriction of Q to the first i columns. Then we can construct a sequence $\mathbb{P}_1, \ldots, \mathbb{P}_n$ such that $\mathbb{P}_i \in \mathcal{N}_{k_i}^{(0,j,\ldots,j)}((d_1, j - d_2, \ldots, j(i-1) - d_i)^{(0,j,\ldots,j)}$ where k_i is the number of rooks in Q_i and $I(\mathbb{P}_i) = \mathbb{P}_i$ as follows.

Case 1. If Q_1 is the empty placement, then $\mathbb{P}_1$ is the empty placement.

Case 2. If Q_i has a rook in row i so that its weight is q^{d-1-i}, then $\mathbb{P}_1$ has rook in row i so that its weight is q^{d_1-i}.

Assuming that we have constructed $\mathbb{P}_i$ so that

$$\tilde{W}^{(j)}_{1,q,F(d_1,\ldots,d_i)}(Q_i) = q^{A_1+\cdots+A_{i-k_i}-\sum_{s=0}^{i} j(s-1)-d_s}\mu_{q,(d_1,j-d_2,\ldots,,j(i-1)-d_i)^{(0,j,\ldots,j)}}(\mathbb{P}_i),$$

we define $\mathbb{P}_{i+1}$ as follows:

Figure 3.3: An example of the bijection Θ between a placement of 3 rooks in a 2-attacking Ferrers board $B = F(0,1,2,3,6)$ and a placement in the corresponding board $\mathcal{B}^{\mathcal{A}}$.

Case 3. If Q_{i+1} has no rook in the last column, then $\mathbb{P}_{i+1}$ is the placement that results by using $\mathbb{P}_i$ in the first i-columns and having no rook in column $i+1$.

Case 4. If Q_{i+1} has a rook s row from the top which contains a cell which is not cancelled so that $\tilde{W}^{(j)}_{1,q,F(d_1,\ldots,d_i,d_{i+1})}(Q_{i+1}) = q^{s-1}$, then $\mathbb{P}_{i+1}$ is the results of starting with $\mathbb{P}_i$ and placing rook in row $(j(i-1)-d_i)+s$ of the augmented part of column i so that weight of $\mathbb{P}_{i+1}$ is just q^{s-1} times the corresponding weight of $\mathbb{P}_i$.

Then we let $\Theta(Q) = \mathbb{P}_n$.

For example, in Figure 3.3, we have pictured $Q = Q_5, \ldots, Q_1$ for a $Q \in \mathcal{N}^{(2)}_3(D)$ where D is the 2-attacking board $F(0,1,2,3,6)$ at the top of the figure. In this case, $\mathcal{B} = (0,1,2,3,2)$ and $\mathcal{A} = (0,2,2,2,2)$. Since we are assuming that $sgn(i) = \overline{sgn}(i)$ for all i, all columns are negative so that we have shaded the cells in the upper augmented board of $\mathcal{B}^{\mathcal{A}}$ which are in the mirror image of its column. Then we have pictured the corresponding sequence of rook placements $P_1, \ldots, P_5$ at the bottom. In each case, we have used a"$\bullet$" to indicate squares cancelled by the rook in column 3 and an "$*$" to indicate the cells cancelled by the rook in column 4.

3.3.2 Case II: $sgn(i) = +1, \overline{sgn}(i) = -1$

Now if $sgn(i) = 1$, then the left-hand side of our q-analogue of the the general product (3.1) formula is $\prod_{i=1}^{k}([x]_q + [b_i]_q)$. Since $[x+b_i]_q = [x]_q + q^x[b_i]_q$, we can can replace by $\prod_{i=1}^{k}([x]_q + [b_i]_q)$ by $\prod_{i=1}^{k}[x+b_i]_q$ on the left-hand side of (3.1) by simply weighting each rook that appears in the $\mathcal{B}$-part of the board $\mathcal{B}^{\mathcal{A}}$ with an extra factor of q^x. Thus, if for any $\mathbb{P} \in \mathcal{NA}_k(\mathcal{B}^{\mathcal{A}})$ we let

$$\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = q^{\mathcal{B}(\mathbb{P})x}\mu_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) \tag{3.20}$$

where $\mathcal{B}(\mathbb{P})$ is the number of rooks of $\mathbb{P}$ which lie in $\mathcal{B}$-part of the board $\mathcal{B}^{\mathcal{A}}$ and if we define

$$\overline{r}^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, q) = \sum_{\mathbb{P}\in\mathcal{N}^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}})} q^{A_1+\cdots+A_k}\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}), \tag{3.21}$$

then one can show, using essentially the same proof that we used to prove (3.1), that

$$\prod_{i=1}^{n}([x+b_i]_q) = \sum_{k=0}^{n}\overline{r}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}}, q)([x-A_1]_q)([x-A_2]_q)\cdots([x-A_k]_q) \tag{3.22}$$

Now suppose that we consider the j-attacking Remmel-Wachs model where we have a j-attacking Ferrers board $D = F(d_1, \ldots, d_n)$ where $d_i \geq j(i-1)$ for all i. In this case, we would be led to the following product formula in the Remmel-Wachs model.

$$\prod_{i=1}^{n}[x+d_i-j(i-1)]_q = \sum_{k=0}^{n}\tilde{r}^{(j)}_{n-k,D}(1,q)[x]_q\downarrow_k. \tag{3.23}$$

To obtain the corresponding product formula from (3.22), we must let $\mathcal{B} = (d_1, d_2-j, \ldots, d_n-j(n-1))$, $\mathcal{A} = (0, j, \ldots, j)$, $sgn(i) = 1$ and $\overline{sgn}(i) = -1$ for all i. In that case, (3.22) becomes

$$\prod_{i=1}^{n}[x+d_i-j(i-1)]_q = \sum_{k=0}^{n}\overline{r}^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, q)[x]_q\downarrow_k. \tag{3.24}$$

In this case, however, the extra factors of q^x that appear in $\overline{r}^{\mathcal{A}}_{k}(\mathcal{B}^{\mathcal{A}}, q)$ do not allow us to conclude that $\overline{r}^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, q) = \tilde{r}^{(j)}_{n-k,D}(1,q)$ for all k. Indeed, it is not that case that $\overline{r}^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, q) = \tilde{r}^{(j)}_{n-k,D}(1,q)$ as the following example will show. Let $j = 2$ and $D = F(1,2)$. Then $\mathcal{B} = (1,0)$ and $\mathcal{A} = (0,2)$. In Figure 3.4, we have pictured all the rook placements in $\mathcal{N}^{(2)}_k(D)$ for $k = 1,2$ and their corresponding weights $\tilde{W}^{(2)}_{1,q,D}(\mathbb{P})$ at the top of the figure. Note $\mathcal{N}^{(2)}_2(D)$ is empty since the rook in first

Figure 3.4: Rook placements in $\mathcal{N}_k^{(2)}(F(1,2))$ and $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ for $k = 1, 2$.

column 2-attacks both cells in the second column and hence $\tilde{r}_{2,D}^{(2)} = 0$. Similarly, at the bottom of the figure, we have pictured all the weights of $\mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ for $k = 1, 2$ and their corresponding weights $q^{A_1+\cdots+A_k}\overline{\mu}_{q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$. Again $\mathcal{N}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ is empty so that $\overline{r}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 0$ It follows that $\tilde{r}_{0,D}^{(2)} = q^3$, $\tilde{r}_{1,D}^{(2)} = 1 + q + q^2$, and $\tilde{r}_{2,D}^{(2)} = 0$ so that (3.22) becomes

$$[x+1]_q[x]_q = q^3[x]_q[x-2]_q + (1+q+q^2)[x]_q. \tag{3.25}$$

Similarly $\overline{r}_0^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 1$, $\overline{r}_1^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 1 + q + q^x$, and $\overline{r}_2^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q) = 0$ so that (3.24) becomes

$$[x+1]_q[x]_q = [x]_q[x-2]_q + (1+q+q^x)[x]_q. \tag{3.26}$$

Note that in this case, these two identities hold because there are two ways to write $[x+1]_q$ when $x \in \mathbb{N}$, namely,

$$[x+1]_q = q^3[x-2]_q + (1+q+q^2) = q^2[x-2]_q + (1+q+q^x). \tag{3.27}$$

3.3.3 Case III: $sgn(i) = -1, \overline{sgn}(i) = +1$

For $x, c \in \mathbb{N}$, we have that $[x]_q + q^x[c]_q = [x+c]_q$. Thus if we want to replace $[x]_q + [A_i]_q$ by $[x + A_i]_q = [x]_q + q^x[A_i]$, then we should weight each rook that lies in upper augmented part of $\mathcal{B}^{\mathcal{A}}$ by an extra factor of q^x. This means that when we consider placements in $\mathcal{B}_x^{\mathcal{A}}$, then we must also weight each rook that lies in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}$ with an extra factor of q^x so that for any given column the weights of possible placements in the lower and upper augmented parts cancel each other as in the proofs in Chapter 2. Thus we define $\tilde{\tilde{r}}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)$ to be the q-weight over all placements of k rooks in $\mathcal{B}^{\mathcal{A}}$ where each rook placed in the augmented part receives an extra factor of q^x. Then it is not difficult to see that we can use essentially the same proof as in Chapter 2 to prove that

$$\prod_{i=1}^{n}([x]_q - [b_i]_q) = \sum_{k=0}^{n} \tilde{\tilde{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)[x + A_1]_q \cdots [x + A_n - k]_q. \tag{3.28}$$

Finally we replace $\tilde{\tilde{r}}_k^{\mathcal{A}}(\mathcal{B}, q)$ by a new q-rook number, $\tilde{\tilde{\tilde{r}}}_k^{\mathcal{A}}(\mathcal{B}, q)$ where

$$\tilde{\tilde{\tilde{r}}}_k^{\mathcal{A}}(\mathcal{B}, q) := q^{-(b_1+\cdots+b_n)}\tilde{\tilde{r}}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q).$$

Then we obtain the following formula:

$$\prod_{i=1}^{n}([x - b_i]_q) = \sum_{k=0}^{n} tilde\tilde{\tilde{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}, q)([x + A_1]_q)([x + A_2]_q) \cdots ([x + A_k]_q). \tag{3.29}$$

3.3.4 Case IV: $sgn(i) = \overline{sgn}(i) = +1$

By combining the ideas of the previous subsections, we can weight each rook in $\mathcal{B}$-part of the board by an extra factor of q^x and weight each rook in both the lower and upper augmented part of the board $\mathcal{B}_x^{\mathcal{A}}$ by an extra factor of q^x to obtain a new q-rook numbers $\overline{\overline{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)$ so that we obtain the following formula:

$$\prod_{i=1}^{n}([x + b_i]_q) = \sum_{k=0}^{n} \overline{\overline{r}}_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, q)([x + A_1]_q)([x + A_2]_q) \cdots ([x + A_k]_q) \tag{3.30}$$

3.3.5 General Modified q-Product Formulas

In this section, we shall make a few remarks about how one can modify our q-analogue of the general product formulas to obtain the following q-analogue of our general product formula

$$\prod_{i=1}^{n}[x + sgn(i)b_i]_q = \sum_{k=0} R_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) \prod_{j=1}^{k}([x + \sum_{s \le j} \overline{sgn}(s)a_s]_q). \tag{3.31}$$

To obtain the left-hand side is easy. That is if $sgn(i) = 1$, then we can obtain a factor of $[x]_q + q^x[b_i]_q = [x + b_i]_q$ by simple weight each cell in the $\mathcal{B}$ part of column i with an extra factor of q^x. If $sgn(i) = -1$, then we keep things the same and we obtain $[x]_q - [b_i]_q = q^{b_i}[x - b_i]_q$.

To obtain the right-hand side, we need only ensure that we weight the cells in the lower augmented part of the board so that the sum of the weights of the cells that lie in a_s part of the lower augmented board for $s \leq i$ is $q^x[\sum_{s=0}^{i} \overline{sgn}(s)a_s]_q$ if $\sum_{s=0}^{i} \overline{sgn}(s)a_s \geq 0$ and is $[\sum_{s=0}^{i} \overline{sgn}(s)a_s]_q$ if $\sum_{s=0}^{i} \overline{sgn}(s)a_s < 0$. We can accomplish this with the following weighting scheme.

To define the weight of a cell c, $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$, we proceed as follows:

1. For each i, the weights $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$ of the cells in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $1, q, q^2, \ldots, q^{x-1}$, reading from bottom to top.

2. For each i, the weights $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$ of the cells in the i-th column of the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$ are -1,$-q$, $-q^2$, $\ldots$, $-q^{b_i-1}$, reading from bottom to top if $sgn(i) = -1$.

3. For each i, the weights $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$ of the cells in the i-th column of the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$ are q^x,q^{x+1}, q^{x+2}, $\ldots$, q^{x+b_i-1}, reading from bottom to top if $sgn(i) = 1$.

4. For each i, we assign weight $M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c)$ to cells in the i-th column of the lower augmented part as follows. First, if $\overline{sgn}(1) = -1$, we assign the weight $-1, -q, -q^2, \ldots, -q^{a_i-1}$ to cells in a_1-st part of column i in the lower augmented board reading from top to bottom. If $\overline{sgn}(1) = 1$, we assign the weight $q^x, q^{x+1}, q^{x+2}, \ldots, q^{x+a_1-1}$ to cells in a_1-st part of column i in the lower augmented board reading from top to bottom.

   Thus the sum of the q-weights of cells in a_1-st part of column i in the lower augmented board is $-[a_1]_q$ if $sgn(1) = -1$ and $q^x[a_1]_q$ if $sgn(1) = 1$. Next suppose that we have assigned the weights to cells in a_j-th part of column i

in the lower augmented part for $j = 1, \ldots s$ so that the sum of the q-weights of cells that lie in a_j-th part of column i in the lower augmented board for $j \leq s$ is $-[\sum_{r=0}^{s} \overline{sgn}(r)a_r]_q$ if $\sum_{r=0}^{s} \overline{sgn}(r)a_r < 0$ and is $q^x[\sum_{r=0}^{s} \overline{sgn}(r)a_r]_q$ if $\sum_{r=0}^{s} \overline{sgn}(r)a_r \geq 0$.

Then we defined the weights to the cells in a_{s+1}-th part of column i in the lower augmented part according to the following cases.

Case 1. $0 \leq \sum_{r=0}^{s} \overline{sgn}(r)a_r \leq \sum_{r=0}^{s+1} \overline{sgn}(r)a_r$. In this case, we assign the weight of cells in the a_{s+1} part to be $q^x q^{\sum_{r=0}^{s} \overline{sgn}(r)a_r}, q^x q^{1+\sum_{r=0}^{s} \overline{sgn}(r)a_r}$, …, $q^x q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)-1}$, reading from top to bottom,

Case 2. $0 \leq \sum_{r=0}^{s+1} \overline{sgn}(r)a_r < \sum_{r=0}^{s} \overline{sgn}(r)a_r$. In this case, we will assign the weight of cells in the a_{s+1} part to be $-q^x q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-1}$, $-q^x q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-2}$, …, $-q^x q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)}$, reading from top to bottom.

Case 3. $(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r) < 0 \leq (\sum_{r=0}^{s} \overline{sgn}(r)a_r)$. In this case, we assign the weight of cells in the a_{s+1} part to be $-q^x q^{\overline{A}_s-1}$, $-q^x q^{\overline{A}_s-2}$, …, $-q^x$,$-q^0$, $-q^1$, …, $-q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)-1}$, reading from top to bottom.

Case 4. $0 \geq (\sum_{r=0}^{s} \overline{sgn}(r)a_r) \geq (\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)$. In this case, we assign the weight of cells in the a_{s+1} part to be $-q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)}, -q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)+1}$, …, $-q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)-1}$, reading from top to bottom,

Case 5. $0 \geq (\sum_{r=0}^{s+1} \overline{sgn}(r)a_r) > (\sum_{r=0}^{s} \overline{sgn}(r)a_r)$. In this case, we assign the weight of cells in the a_{s+1} part to be $q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-1}, q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-2}$, …, $q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)}$, reading from top to bottom.

Case 6. $(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r) > 0 \geq (\sum_{r=0}^{s} \overline{sgn}(r)a_r)$. In this case, we assign the weight of cells in the a_{s+1} part to be $q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-1}, q^{(\sum_{r=0}^{s} \overline{sgn}(r)a_r)-2}$, …, q, q^x, q^{x+1}, …, $q^x q^{(\sum_{r=0}^{s+1} \overline{sgn}(r)a_r)-1}$, reading from top to bottom.

5. For each i, the cell r-th row of the i-th column of the upper augmented, reading from bottom to top, is equal to -1 times the weight of the cell in

Figure 3.5: The modified q-weighting of cells in placements in $\mathcal{B}_x^{\mathcal{A}}$.

r-row of i-th column of the lower augmented board, reading from top to bottom. That is, in the upper augmented part of column i, the weight of a cell in the i-th column is in opposing sign to that cell's exact mirror image in the lower augmented part.

An example of this kind of labeling can be seen in the left-hand side of Figure 3.5 where $x = 4$, $\mathcal{B} = (1,2,2,4)$, $\mathcal{A} = (2,1,2,1)$,

$$sgn(i) = \begin{cases} +1 & \text{if } i = 1,2,4, \\ -1 & \text{if } i = 3 \end{cases}$$

and

$$\overline{sgn}(i) = \begin{cases} +1 & \text{if } i = 2,3,4, \\ -1 & \text{if } i = 1 \end{cases}$$

Now suppose that $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ has rooks in cell $c_1, \ldots, c_k$. Then we set

$$M_{q,sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{k} M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{3.32}$$

Similarly, if $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_n$, we will let

$$M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) = \prod_{i=1}^{n} M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{3.33}$$

Then we define

$$MR_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} M_{q,sgn,\overline{sgn},\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{3.34}$$

Then by computing the sum

$$H(q) = \sum_{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} M_{q,sgn,\overline{sgn},\mathcal{B}_x^{\mathcal{A}}}(Q) \tag{3.35}$$

in two different ways as we did in the proof of (3.1), we can prove the following:

$$
\begin{aligned}
&(\prod_{i:sgn(i)=1}([x]_q+q^x[b_i]_q))(\prod_{i:sgn(i)=-1}([x]_q-[b_i]_q)) && (3.36)\\
&= \sum_{k=0} MR^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q)\prod_{i=1}^{k}\phi(i)
\end{aligned}
$$

where

$$
\phi(s)=\begin{cases}[x]_q+q^x[\sum_{r=0}^{s}\overline{sgn}(r)a_r]_q & \text{if } (\sum_{r=0}^{s}\overline{sgn}(r)a_r)\geq 0, \text{ and}\\ [x]_q-[\sum_{r=0}^{s}\overline{sgn}(r)a_r]_q & \text{if } (\sum_{r=0}^{s}\overline{sgn}(r)a_r)<0\end{cases} \qquad (3.37)
$$

Then if we set

$$
\begin{aligned}
R^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q) &= (\prod_{\substack{r\leq k\\ \sum_{r=0}^{s}\overline{sgn}(r)a_r<0}} q^{(\sum_{r=0}^{s}\overline{sgn}(r)a_r)}) MR^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, q)\\
&= \times(\prod_{i:sgn(i)=-1} q^{-b_i}), \qquad (3.38)
\end{aligned}
$$

then we will have the following theorem.

Theorem 3.2. *For all sequences of non-negative integers, $\mathcal{B}=(b_1,\ldots,b_n)$ and $\mathcal{A}=(a_1,\ldots,a_n)$, and functions $sgn, \overline{sgn}:\{1,\ldots,n\}\to\{1,-1\}$,*

$$
\prod_{i=1}^{n}[x+sgn(i)b_i]_q=\sum_{k=0} R^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, q)\prod_{j=1}^{k}([x+\sum_{s\leq j}\overline{sgn}(s)a_s]_q). \qquad (3.39)
$$

3.4 A (P, Q)-Analogue of the General Product Formula

In this section, we will define an appropriate p, q-analogue of the rook numbers $r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn})$, $r^{\mathcal{A}}_{n-k}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, p, q)$, so that we can prove the following theorem.

Theorem 3.3. *Suppose $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are two sequences of nonnegative integers, and let $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$ be two sign functions. Then,*

$$\prod_{i=1}^{n}([x]_{p,q} + sgn(i)[b_i]_{p,q}) = \sum_{k=0}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, p, q) \prod_{s=1}^{k}([x]_{p,q} + [\overline{A}_s]_{p,q}). \quad (3.40)$$

Here we use the convention that if n is positive integer, then $[-n]_{p,q} = -[n]_{p,q}$, and we will refer to Equation (3.40) as the *p, q-general product formula.*

The proof of this theorem is essentially the same as the proof of Theorem (3.1) with the exception that we have to use a different weighting function on the cells. However, this case is a bit harder because we cannot just have the p, q weight of the cell be of the form $\pm p^a q^b$ in the lower augmented part of the board. That is, the key property of our q-weighting of cells is that the sum of the q-weights of the cells that lie in a_j-part of column i in the lower augmented part of the board for $j \leq s$ was $[\overline{A}_s]_q$ and the sum of q-weights of the cells that lie in a_j-part of column i in the upper augmented part of the board for $j \leq s$ was $-[\overline{A}_s]_q$. We would like to define the p, q-weights of the cells so that the sum of the p, q-weights of the cells that lie in a_j-part of column i in the lower augmented part for $j \leq s$ is $[\overline{A}_s]_{p,q}$ and the sum of q-weights of the cells that lie in a_j-part of column i in the upper augmented part for $j \leq s$ is $-[\overline{A}_s]_{p,q}$. Now suppose that $sgn(1) = sgn(2) = 1$ and $a_1 = a_2 = 3$. Then the most natural thing to do would be to assign the p, q-weights to the cells in the a_1-st part of the lower augmented board to be p^2, pq, q^2 reading from top to bottom. However, at that point, we want the sum of the p, q-weights of the cells that lie in a_1-st part plus a_2-nd part of column i to be $[6]_{p,q} = p^5 + p^4q + p^3q^2 + p^2q^3 + pq^4 + p^5$. But there is no way to weight the cells of the a_2-nd part with weights of the form $p^a q^b$ to transform $[3]_{p,q}$ to $[6]_{p,q}$. Thus we have to allow the p, q-weights of cells to be polynomials in p and q if we are going to be able to make such a transformation. Our idea is quite simple. Namely, we shall just weight the lowest cell of the a_2-nd part with $[6]_{p,q} - [3]_{p,q}$ and the other cells with 0. Extending this idea will allow us to define the p,q-weights of the cells so that the sum of the p,q-weights of the cells that lie in a_j-part of column i in the lower augmented board for $j \leq s$ is $[\overline{A}_s]_{p,q}$ and the sum of q-weights of the cells that lie in a_j-part of column i in the upper augmented board for $j \leq s$ is $-[\overline{A}_s]_{p,q}$.

Fix the two sequences $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ and the two sign functions $sgn : \{1, \ldots, n\} \to \{1, -1\}$ and $\overline{sgn} : \{1, \ldots, n\} \to \{1, -1\}$. The first step in proving Equation (3.40) is to define a p,q-weight, $\mu_{p,q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P})$, of each placement $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ and a p,q-weight, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(Q)$, to each placement $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$. To do this, we shall define a p,q-weight, $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, to each cell c in $\mathcal{B}_x^{\mathcal{A}}$. Then if $\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_k$, we set

$$\mu_{p,q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}) = \prod_{i=1}^{k} \mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{3.41}$$

Similarly, if $Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})$ has rooks in cells $c_1, \ldots, c_n$, then

$$\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(Q) = \prod_{i=1}^{n} \mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c_i). \tag{3.42}$$

Then we define

$$r_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}, sgn, \overline{sgn}, p, q) = \sum_{\mathbb{P} \in \mathcal{N}_k^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}})} \mu_{p,q,\mathcal{B}^{\mathcal{A}}}(\mathbb{P}). \tag{3.43}$$

To define $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$, we proceed as follows:

1. For each i, the weights $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$ of the cells in the i-th column of the x-part of $\mathcal{B}_x^{\mathcal{A}}$ are $p^{x-1}, p^{x-2}q, p^{x-3}q^2, \ldots, q^{x-1}$, reading from bottom to top.

2. For each i, the weights $\mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(c)$ of the cells in the i-th column of the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A}}$ are $sgn(i)p^{b_i-1}$, $sgn(i)p^{b_i-2}q$, $sqn(i)p^{b_i-3}q^2$, $\ldots, sgn(i)q^{b_i-1}$, reading from bottom to top.

3. For each i, we assign weight $\mu_{p,q,\mathcal{B}_x^A}(c)$ to cells in the i-th column of the lower augmented part as follows. First, we assign the weight $\overline{sgn}(i)p^{a_1-1}$, $\overline{sgn}(i)p^{a_1-2}q$, $\overline{sgn}(i)p^{a_1-3}q^2$, …, $\overline{sgn}(i)q^{a_1-1}$ to the cells in a_1-st part of column i in the lower augmented part of the board reading from top to bottom. Thus the sum of the q-weights of cells in a_1-st part of column i in the lower augmented part is $[\overline{sgn}(i)a_1]_{p,q}$. Next suppose that we have assigned the weights to cells in a_j-th part of column i in the lower augmented part for $j = 1, \ldots, s$ so that the sum of the p, q-weights of cells that lie in a_j-th part of column i in the lower augmented part for $j \leq s$ is $[\overline{A}_s]_{p,q}$. Then we define the weights to the cells in a_{s+1}-th part of column i in the lower augmented part according to the following cases:

Case 1. $0 \leq \overline{A}_s \leq \overline{A}_{s+1}$. In this case, we assign the weight of cells in the a_{s+1} part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom.

Case 2. $0 \leq \overline{A}_{s+1} < \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom.

Case 3. $\overline{A}_{s+1} < 0 \leq \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $-[\overline{A}_s]_{p,q}, 0, \ldots, 0, -p^{|\overline{A}_{s+1}|-1}, -qp^{|\overline{A}_{s+1}|-2}, \ldots - q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

Case 4. $0 \geq \overline{A}_s \geq \overline{A}_{s+1}$. In this case, we assign the weight of cells in the a_{s+1} part to be $[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$, reading from top to bottom,

Case 5. $0 \geq \overline{A}_{s+1} > \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be

$$[\overline{A}_{s+1}]_{p,q} - [\overline{A}_s]_{p,q}, 0, \ldots, 0$$

, reading from top to bottom.

Case 6. $\overline{A}_{s+1} > 0 \geq \overline{A}_s$. In this case, we assign the weight of cells in the a_{s+1} part to be $-[\overline{A}_s]_{p,q}, 0, \ldots, 0, p^{|\overline{A}_{s+1}|-1}, qp^{|\overline{A}_{s+1}|-2}, \ldots q^{|\overline{A}_{s+1}|-1}$, reading from top to bottom.

4. For each i, the cell in the r-th row of the i-th column of the upper augmented part of the board, reading from bottom to top, is equal to -1 times the weight of the cell in r-row of i-th column of the lower augmented part of the board, reading from top to bottom. That is, in the upper augmented part of column i, the weights of the cells in the i-th column are the exact mirror image as those which come from the lower augmented part.

Then we can prove Theorem 3.3 by computing the sum

$$\sum_{Q \in \mathcal{N}_n^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}})} \mu_{p,q,\mathcal{B}_x^{\mathcal{A}}}(Q) \tag{3.44}$$

in two different ways as before.

We can also obtain a variations of the p,q-analogue Theorem 3.3 where we replace $[x]_{p,q} - [a]_{p,q}$ by $[x-a]_{p,q}$ and replace $[x]_{p,q} + [a]_{p,q}$ by $[x+a]_{p,q}$ much as we did in the q-anlogue case by using the transformations

$$[x]_{p,q} - p^{x-a}[a]_{p,q} = q^a[x-a]_{p,q} \tag{3.45}$$

and

$$p^a[x]_{p,q} + q^x[a]_{p,q} = q^a[x+a]_{p,q} \tag{3.46}$$

where $x \geq a \geq 0$.

Chapter 4

Poly-Stirling Numbers

4.1 Introduction

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m \in \mathbb{N}[x]$. Then we define the $p(x)$-Stirling numbers of of the first and second kind by the following recursions:

$$\begin{aligned} & s_{0,0}^{p(x)} = 1 \text{ and } s_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n\text{, and} \\ & s_{n+1,k}^{p(x)} = s_{n,k-1}^{p(x)} - p(n) s_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} & S_{0,0}^{p(x)} = 1 \text{ and } S_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n\text{, and} \\ & S_{n+1,k}^{p(x)} = S_{n,k-1}^{p(x)} + p(k) S_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.2}$$

It turns out that for any $p(x) \in \mathbb{N}[x]$, the matrices $||s_{n,k}^{p(x)}||$ and $||S_{n,k}^{p(x)}||$ are inverses of each other. This fact follows from the following general inversion formula of Milne [22].

Theorem 4.1. *Let $n, k \in \mathbb{N}$ and suppose $\{x_j\}$ is any sequence indexed by $\mathbb{N}$. Set $||A||$ to be the upper triangular matrix whose $(i+1, j+1)^{st}$ entry, $A_{i+1,j+1} = a_{i,j}$, where the numbers $a_{n,k}$ satisfy the recursions*

$$a_{0,0} = 1 \textit{ and } a_{n,k} = 0 \textit{ if } k < 0 \textit{ or } k > n\textit{, and}$$
$$a_{n+1,k} = a_{n,k-1} + x_k a_{n,k} \textit{ if } 0 \leq k \leq n+1 \textit{ and } n \geq 0.$$

Then the matrix $||B||$*, whose* $(i+1, j+1)^{st}$ *entry ,* $B_{i+1,j+1} = b_{i,j}$*, where the numbers* $b_{n,k}$ *satisfy the recursions*

$$b_{0,0} = 1 \textit{ and } b_{n,k} = 0 \textit{ if } k < 0 \textit{ or } k > n\textit{, and}$$
$$b_{n+1,k} = b_{n,k-1} - x_n b_{n,k} \textit{ if } 0 \leq k \leq n+1 \textit{ and } n \geq 0,$$

is the inverse of $||A||$*.*

If we replace $s_{n,k}^{p(x)}$ with $(-1)^{n-k} c_{n,k}^{p(x)}$, then we have the recursions:

$$c_{0,0}^{p(x)} = 1 \text{ and } c_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n\text{, and} \tag{4.3}$$
$$c_{n+1,k}^{p(x)} = c_{n,k-1}^{p(x)} + p(n) c_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

We will call the sequence of numbers in Equation (4.2) the *poly-Stirling numbers of the second kind* for $p(x)$, the sequence of numbers in Equation (4.1) the *poly-Stirling numbers of the first kind* for $p(x)$, and the sequence of numbers in Equation (4.3) the *signless poly-Stirling numbers of the first kind* for $p(x)$.

In this chapter we will give rook theoretic interpretations of these poly-Stirling numbers of the first and second kind. We shall also show that there are two natural q-analogues of these numbers depending on whether we take the q-analogue of $p(n)$ to be $p([n]_q)$ or $[p(n)]_q$. That is, if we take the q-analogue of $p(n)$ to be $p([n]_q)$, then we can define q-analogues of $s_{n,k}^{p(x)}$ and $S_{n,k}^{p(x)}$ by the following recursions:

$$s_{0,0}^{p(x)}(q) = 1 \text{ and } s_{n,k}^{p(x)}(q) = 0 \text{ if } k < 0 \text{ or } k > n\text{, and} \tag{4.4}$$
$$s_{n+1,k}^{p(x)}(q) = s_{n,k-1}^{p(x)}(q) - p([n]_q) s_{n,k}^{p(x)}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

and

$$S_{0,0}^{p(x)} = 1 \text{ and } S_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \qquad (4.5)$$
$$S_{n+1,k}^{p(x)} = S_{n,k-1}^{p(x)} + p([k]_q) S_{n,k}^{p(x)} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

If we take the q-analogue of $p(n)$ to be $[p(n)]_q$, then we can define q-analogues of $s_{n,k}^{p(x)}$ and $S_{n,k}^{p(x)}$ by the following recursions:

$$\overline{s}_{0,0}^{p(x)}(q) = 1 \text{ and } \overline{s}_{n,k}^{p(x)}(q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \qquad (4.6)$$
$$\overline{s}_{n+1,k}^{p(x)}(q) = \overline{s}_{n,k-1}^{p(x)}(q) - [p(n)]_q \overline{s}_{n,k}^{p(x)}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

and

$$\overline{S}_{0,0}^{p(x)}(q) = 1 \text{ and } \overline{S}_{n,k}^{p(x)}(q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \qquad (4.7)$$
$$\overline{S}_{n+1,k}^{p(x)}(q) = \overline{S}_{n,k-1}^{p(x)}(q) + [p(k)]_q \overline{S}_{n,k}^{p(x)}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

We shall call the first type of q-analogues, the type I q-analogue of $s_{n,k}^{p(x)}$ and $S_{n,k}^{p(x)}$ and the second type of q-analogues, the type II q-analogue of $s_{n,k}^{p(x)}$ and $S_{n,k}^{p(x)}$.

The outline of this chapter is as follows. First, we shall present a general rook model, which we call m-partition boards, whose file numbers and rook numbers will, respectively, specialize to the $s_{n,k}^{x^m}$ and $S_{n,k}^{x^m}$ when we restrict ourselves to the analogue of the staircase board. Then we shall give two different q-analogues of our model's file and rook numbers. In one case, the file and rook numbers for the staircase board will specialize to the $s_{n,k}^{x^m}(q)$ and $S_{n,k}^{x^m}(q)$ and, in the other case, the file and rook numbers for the staircase board will specialize to the $\overline{s}_{n,k}^{x^m}(q)$ and $\overline{S}_{n,k}^{x^m}(q)$. Then we will show how we can combine these various x^m-models to obtain a rook theory model which will allows us to give a rook theory interpretation for $s_{n,k}^{p(x)}$ and $S_{n,k}^{p(x)}$ and their q-analogues for any polynomial $p(x) \in \mathbb{N}[x]$.

Figure 4.1: $B^{(2)}$, with $B = F(0,1,3,4)$

4.2 m-Partition Boards & Rook Placements

Let $B = F(b_1, b_2, \ldots, b_n)$ be a rook board and suppose that m is a positive integer. We may then define $B^{(m)}$, called the *m-partition of B*, to be the board B where each column is partitioned into m subcolumns. We will define, for any board A, $C_{(l,j)}(A)$ to be the l^{th} subcolumn, from left to right, of the j^{th} column of A. An example of these types of boards can be seen in Figure 4.1, where $B = F(0,1,3,4)$ and $m = 2$. We also define the n^{th} *m-staircase board*, denoted by $\mathbf{B}_n^{(m)}$, to be the n^{th} staircase board, $\mathbf{B}_n$, with each column partitioned into m subcolumns.

As in Chapter 1, we can define two kinds of rook placements in the board $B^{(m)}$: non-attacking placements and file placements. In order to do this, we must make the rule that if one rook is placed in any one subcolumn of $B^{(m)}$, then a rook must be placed in every other subcolumn of that same column, that is, if a rook lies in a cell of $C_{(l,j)}(B^{(m)})$ for some l and j, then a rook lies in a cell of $C_{(s,j)}(B^{(m)})$ for each $1 \leq s \leq m$. For notational purposes, if m rooks are placed in the j^{th} column of $B^{(m)}$, then those rooks will be denoted by the m-tuple $r^{(j,m)} = (r_{(1,j)}, r_{(2,j)}, \ldots, r_{(m,j)})$, interpreted as there being a rook $r_{(l,j)}$ which is placed in $C_{(l,j)}(B^{(m)})$. Moreover, if the rook $r_{(l,j)}$ is placed in the cell of $C_{(l,j)}(B^{(m)})$ which is in the i^{th} row from the bottom of $B^{(m)}$, then we will say that the rook $r_{(l,j)}$ is in cell $c(i,l,j)$. Finally, for a rook placement which has rooks in k columns of $B^{(m)}$, we will define C_{i_w} to be the w^{th} column of $B^{(m)}$, from left to right, which contains a rook, and the height of this column is given by b_{i_w}.

Suppose that $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board. For any rook placement in $B^{(m)}$, a rook will cancel the cells in $B^{(m)}$ which are below it and the cells which lie to its right in its respective subcolumn. Formally, if a rook $r = r_{(l,j)}$ is placed in $c(i,l,j)$, then r will cancel the cells $\{c(t,l,j) : 1 \leq t < i\} \cup \{c(i,l,s) : j <$

Figure 4.2: A placement in $\mathcal{N}_{3,(2)}(B^{(2)})$, with $B = F(1,3,3,5,6)$

Figure 4.3: A placement in $\mathcal{F}_{3,(2)}(B^{(2)})$, with $B = F(1,3,3,5,6)$

$s \leq n\}$ which have not been previously cancelled by a rook placed to their left. An example of this cancellation can be seen in Figure 4.2, where the number $''w''$ in the diagram represents a cell cancelled by a rook in column C_{i_w}. We let $\mathcal{N}_{k,(m)}(B^{(m)})$ denote the set of placements of mk rooks in $B^{(m)}$ such that there are exactly k columns which contain rooks, there is at most one rook in any subcolumn, and no rook lies in a cell which is cancelled by a rook to its left. We call such placements, *non-attacking rook placements*.

For any rook board B, we let $\mathcal{F}_{k,(m)}(B^{(m)})$ denote the set of rook placements mk rooks so that there are exactly k columns of $B^{(m)}$ which contain a rook and there is at most one rook in any given subcolumn. We call such a rook placement, a *file rook placement*. In a file placement, each rook $r_{(l,j)}$ placed in cell $c(i,l,j)$ will cancel the cells $\{c(t,l,j) : 1 \leq t < i\}$. An example of this cancellation can be seen in Figure 4.3, where again, the rooks placed in column C_{i_w} cancel the cells marked with a $''w''$.

We then define

$$\begin{aligned} r_{k,(m)}(B^{(m)}) &:= |\mathcal{N}_{k,(m)}(B^{(m)})| \quad \text{and} \\ f_{k,(m)}(B^{(m)}) &:= |\mathcal{F}_{k,(m)}(B^{(m)})|, \end{aligned}$$

and we shall call $r_{k,(m)}(B^{(m)})$ the k^{th} *m-rook number of* $B^{(m)}$ and $f_{k,(m)}(B^{(m)})$ the k^{th} *m-file number of* $B^{(m)}$.

Given a rook board B and a positive integer m, we define the board $B_x^{(m)}$ to be the board $B^{(m)}$ with x rows appended below , each with n columns partitioned into m subcolumns. We refer to this part of the board as the *x-part* and the part that corresponds to $B^{(m)}$ will be called the *upper part* of $B_x^{(m)}$. We will say that these two parts are separated by the *high bar*. An example of this type

Figure 4.4: The board $B_x^{(2)}$, with $B = F(1,3,3,5,6)$ and $x = 4$.

of board can be seen in the Figure 4.4, where $B = F(1,3,3,5,6)$, $m = 2$, and $x = 4$. For $B_x^{(m)}$, we will label the cells in the upper part of this board exactly as we would in the board $B^{(m)}$. For the x-part of $B_x^{(m)}$, we will label the rows, from top to bottom, with $1, 2, \ldots, x$. If a rook r is placed in column $C_{(l,j)}(B_x^{(m)})$ in the x-part in the row labeled with i, then we say that r lies in the cell $c_x(i,l,j)$.

We consider rook placements in the board $B_x^{(m)}$ with the following conditions:

1. m rooks must be placed in every column of $B_x^{(m)}$, with one rook per subcolumn.
2. If any of the m rooks placed in a given column lie above the high bar, then all m rooks in that column must lie above the high bar. Otherwise, all m rooks in that column lie in the x-part.
3. Given a placement of mn rooks in $B_x^{(m)}$, if there are km rook which lie above the high bar, then these rooks form a placement in $\mathcal{F}_{k,(m)}(B^{(m)})$.

We call this type of placement a *file placement in* $B_x^{(m)}$, and we denote the set of all such placements of mn rooks in this board by $\mathcal{F}_{n,(m)}(B_x^{(m)})$. We also make the following cancellation rules for rooks placed in $B_x^{(m)}$:

1. A rook r placed above the high bar in the cell $c(i,l,j)$ will cancel the cells $\{c(t,l,j) : 1 \leq t < i\}$.
2. Rooks placed in the x-part of the board do not cancel any cells.

An illustration of this type of placement and corresponding cancellation can be seen in Figure 4.5, where the cancelled cells are denoted by a "•".

Figure 4.5: A placement in $\mathcal{F}_{5,(2)}(B_x^{(2)})$ and the corresponding cancellation, with $B = F(1,3,3,5,6)$ and $x = 4$.

Theorem 4.2. *Suppose* $x, n \in \mathbb{N}$ *and let* $B = F(b_1, b_2, \ldots, b_n)$ *be a rook board. Then, for any positive integer* m*,*

$$\prod_{i=1}^{n}(x^m + (b_i)^m) = \sum_{k=0}^{n} f_{n-k,(m)}(B^{(m)})(x^m)^k \tag{4.8}$$

Proof: Given a rook board $B = F(b_1, b_2, \ldots, b_n)$, we shall show that Equation (4.8) represents two ways to count $|\mathcal{F}_{n,(m)}(B_x^{(m)})|$. That is, in the standard way, we first consider the number of ways that we can place m rooks in each column, starting with the leftmost column and working to the right. In the first column, there will be x^m cells in which to place the m rooks if we choose to place them in the x-part, and there are $(b_1)^m$ ways of placing them if they all lie in the upper part of the board. This gives us a total of $(x^m + (b_1)^m)$ ways of placing m rooks in the first column of $B_x^{(m)}$. Since rooks in this board do not cancel to their right, we will have, in the i^{th} column of the board, x^m ways to place rooks below the high bar and $(b_i)^m$ ways of placing the m rooks above the high bar, giving a total of $(x^m + (b_i)^m)$ ways of placing m rooks in column i. Thus,

$$|\mathcal{F}_{n,(m)}(B_x^{(m)})| = \prod_{i=1}^{n}(x^m + (b_i)^m).$$

Next, suppose that we first fix a file placement $\mathbb{P}$ with $m(n-k)$ rooks above the bar in $B_x^{(m)}$. We claim that there are $(x^m)^k$ ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_{n,(m)}(B_x^{(m)})$ such that $Q \cap B^{(m)} = \mathbb{P}$. That is, we want to count the number of ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_{n,(m)}(B_x^{(m)})$ by placing an additional mk rooks below the bar. Here, we see that for each empty column, there are

Figure 4.6: The board $\mathcal{B}_x^{\mathcal{A},(2)}$, with $\mathcal{B} = (1,3,3,5,6)$, $\mathcal{A} = (0,1,2,1,2)$, and $x = 4$.

exactly x^m ways to place m rooks below the bar. Thus,

$$\begin{aligned}
|\mathcal{F}_{n,(m)}(B_x^{(m)})| &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{F}_{n-k,(m)}(B^{(m)})} (x^m)^k \\
&= \sum_{k=0}^{n} (x^m)^k \sum_{\mathbb{P} \in \mathcal{F}_{n-k,(m)}(B^{(m)})} 1 \\
&= \sum_{k=0}^{n} (x^m)^k f_{n-k,(m)}(B^{(m)}),
\end{aligned}$$

which is the desired result. □

Now consider the sequences $\mathcal{B} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ and $\mathcal{A} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$. As in Chapter 2, we can define the board $\mathcal{B}^{\mathcal{A}}$ to be the the augmented rook board, and given some $x \geq 0$, the board $\mathcal{B}_x^{\mathcal{A}}$ to be the general augmented rook board. We now define the board $\mathcal{B}_x^{\mathcal{A},(m)}$ by taking the board $\mathcal{B}_x^{\mathcal{A}}$ and partitioning every column into m subcolumns. Here, we will use the terms a_s-part, $\mathcal{B}$-part, x-part, upper augmented part, lower augmented part, high bar, and low bar exactly as in Chapters 2 and 3. An example of this type of board can be seen in Figure 4.6, where $\mathcal{B} = (1,3,3,5,6)$, $m = 2$, $x = 4$, and $\mathcal{A} = (0,1,2,1,2)$. Similarly, the board $\mathcal{B}^{\mathcal{A},(m)}$ will be the board $\mathcal{B}^{\mathcal{A}}$ with each column partitioned into m subcolumns.

If a rook r is placed in the i^{th} row above the high bar in $C_{(l,j)}(\mathcal{B}_x^{\mathcal{A},(m)})$, then we again say that r lies in the cell $c(i,l,j)$. Similarly, if r is placed in the x-part in the row labeled i in $C_{(l,j)}(\mathcal{B}_x^{\mathcal{A},(m)})$, then we say that r lies in the cell $c_x(i,l,j)$. Finally, if a rook r is placed in the lower augmented part in column $C_{(l,j)}(\mathcal{B}_x^{\mathcal{A},(m)})$ in the i^{th} row below the low bar, then we say that r lies in cell $\tilde{c}(i,l,j)$.

First we will consider placements $\mathbb{P}$ of mk rooks in the board $\mathcal{B}^{\mathcal{A},(m)}$ such that if any subcolumn of a column contains a rook, then all the subcolumns of that column must contain a rook. If any rook $r \in \mathbb{P}$ lies in a subcolumn l, then

Figure 4.7: A placement $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{2,(2)}(\mathcal{B}^{\mathcal{A},(2)})$, with $\mathcal{B} = (1,3,3,5,6)$ and $\mathcal{A} = (0,1,2,1,2)$.

it will cancel all cells in the a_s-part of subcolumn l in all columns to its right where $a_s \neq 0$ and a_s is the part of highest index which has not been cancelled by a rook in a column which lies to the left of the column in r. An example of this cancellation can be seen in Figure 4.7, where the cancellation from the rooks placed in the second column is denoted with a "•" and the cancellation from the rooks placed in the fourth column is denoted by an "$*$". Then we let $\mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})$ denote the set of all such placements of km rooks where no rook lies in a cell which is cancelled by a rook to its left. We shall call a placement in $\mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})$ a *non-attacking rook placement* in $\mathcal{B}^{\mathcal{A},(m)}$.

We will now define the k^{th} *type* $\mathcal{A}$ m*-rook number of* $\mathcal{B}^{\mathcal{A},(m)}$ to be

$$r^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)}) = |\mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})|. \tag{4.9}$$

Now consider placements in the board $\mathcal{B}^{\mathcal{A},(m)}_x$. For this board, we make the following placement rules:

1. m rooks must be placed in every column of $\mathcal{B}^{\mathcal{A},(m)}_x$, with one rook per subcolumn.

2. All m rooks placed in a given column lie either completely in the $\mathcal{B}$-part, completely in the upper augmented part, completely in the x-part, or completely below the low bar in $\mathcal{B}^{\mathcal{A},(m)}_x$.

3. Given a placement of mn rooks in $\mathcal{B}^{\mathcal{A},(m)}_x$, if there are km rook which lie above the high bar, then these rooks form a placement in $\mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})$.

4. A rook r placed in the upper part of $\mathcal{B}^{\mathcal{A},(m)}_x$ in the cell $c(i,l,j)$ will cancel above the high bar exactly as in the board $\mathcal{B}^{\mathcal{A},(m)}$. Moreover, if a rook r which lies in column j and subcolumn l cancels the cells in the a_s-part of l^{th} subcolumn in column $t > j$ of the upper augmented board, then it

Figure 4.8: A placement in $\mathcal{N}^{\mathcal{A}}_{5,(2)}(B^{(2)}_x)$ and the corresponding cancellation, with $\mathcal{B} = (1,3,3,5,6)$, $\mathcal{A} = (0,1,2,1,2)$, and $x = 4$.

also cancels the cells a_s-part in the l^{th} subcolumn in column t in the lower augmented part of the board.

5. Rooks which are placed below the low bar do not cancel any cells.

We let $\mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}^{\mathcal{A},(m)}_x)$ denote the set of all placements of nm rooks in $\mathcal{B}^{\mathcal{A},(m)}_x$ satisfying the above conditions and for which no rook lies in a cell which is cancelled by a rook to its left. We call the elements of $\mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}^{\mathcal{A},(m)}_x)$ a *non-attacking placement in* $\mathcal{B}^{\mathcal{A},(m)}_x$. An illustration of this type of placement and corresponding cancellation can be seen in Figure 4.8.

Theorem 4.3. *Given* $x, n \in \mathbb{N}$ *and* $m \in \mathbb{Z}^+$, *let* $\mathcal{B} = (b_1, b_2, \ldots, b_n)$ *and* $\mathcal{A} = (a_1, a_2, \ldots, a_n)$ *be two sequences in* $\mathbb{N}^n$. *Then*

$$\prod_{i=1}^{n}(x^m + (b_i)^m) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k,(m)}(\mathcal{B}^{\mathcal{A},(m)})(x^m - (A_1)^m)(x^m - (A_2)^m)\cdots(x^m - (A_k)^m). \tag{4.10}$$

where $A_i = a_1 + a_2 + \cdots + a_i$.

Proof: Define, for any $Q \in \mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}^{\mathcal{A},(m)}_x)$,

$$W_{x,\mathcal{B},\mathcal{A},(m)}(Q) = \prod_{j=1}^{n} \omega_{x,\mathcal{B},\mathcal{A},(m)}(r^{(j,m)}),$$

where $r^{(j,m)}$ represents the m rooks which are placed in column j and we define $\omega_{x,\mathcal{B},\mathcal{A},(m)}(r^{(j,m)})$ by

1. $\omega_{x,\mathcal{B},\mathcal{A},(m)}(r^{(j,m)}) = -1$ if the m rooks in $r^{(j,m)}$ lie in the lower augmented part of $\mathcal{B}^{\mathcal{A},(m)}_x$ and

2. $\omega_{x,\mathcal{B},\mathcal{A},(m)}(r^{(j,m)}) = 1$ otherwise.

If $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})$, and if the k rooks are in columns $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$, then set

$$W_{\mathcal{B},\mathcal{A},(m)}(\mathbb{P}) = \prod_{j=1}^{k} \omega_{x,\mathcal{B},\mathcal{A},(m)}(r^{(i_j,m)}).$$

If we define

$$S(\mathcal{B}^{\mathcal{A},(m)}_x) = \sum_{\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}^{\mathcal{A},(m)}_x)} W_{x,\mathcal{B},\mathcal{A},(m)}(\mathbb{P}),$$

then we claim that Theorem 4.3 represents computing $S(\mathcal{B}^{\mathcal{A},(m)}_x)$ in two different ways.

First, suppose we wish to place m rooks in every column of $\mathcal{B}^{\mathcal{A},(m)}_x$ by starting with the leftmost column and working to the right. Then in the first column of $\mathcal{B}^{\mathcal{A},(m)}_x$, there are x rows to place a rook in the x-part, and each is divided into m subcolumns, so we have x^m ways to place rooks in the x-part of the first column. We also have b_1 rows to place rooks in the $\mathcal{B}$-part, which gives us a total of $(b_1)^m$ ways to place rooks above the high bar. Finally, we have A_1 rows to place rooks in the both the upper and lower augmented parts of the board. Thus there are A_1^m ways to place the rooks in the upper augmented board which contribute a factor of A_1^m. Similarly, there are A_1^m ways to place the rooks in the lower augmented board which contribute a factor of $-A_1^m$. Thus the total weighting for these placements in the first column will be $(x^m + (b_1)^m + (A_1)^m - (A_1)^m) = (x + (b_1)^m)$. In second column, we have two cases.

<u>Case I:</u> If the m rooks placed in the first column were placed above the high bar, then the cells in the a_2-nd part of the upper and lower augmented boards are cancelled in the second column. Thus, there will be x^m ways of placing rooks in the x-part, $(b_2)^m$ ways of placing rooks in the $\mathcal{B}$-part of $\mathcal{B}^{\mathcal{A},(m)}_x$, $(A_1)^m$ ways of placing rooks in the upper augmented part of $\mathcal{B}^{\mathcal{A},(m)}_x$, and $(A_1)^m$ ways of placing rooks in the lower augmented part of $\mathcal{B}^{\mathcal{A},(m)}_x$. Thus, the total weight over all placements in the second column is of $\mathcal{B}^{\mathcal{A},(m)}_x$ is $(x^m + (b_2)^m + (A_1)^m - (A_1)^m) =$

$(x^m + (b_2)^m)$.

<u>Case II:</u> If the m rooks placed in the first column were placed below the high bar, then no cells in the second column were cancelled. Thus, there will be x^m ways of placing rooks in the x-part, $(b_2)^m$ ways of placing rooks in the $\mathcal{B}$-part of $\mathcal{B}_x^{\mathcal{A},(m)}$, $(A_2)^m$ ways of placing rooks in the upper augmented part of $\mathcal{B}_x^{\mathcal{A},(m)}$, and $(A_2)^m$ ways of placing rooks in the lower augmented part of $\mathcal{B}_x^{\mathcal{A},(m)}$. However, if the m rooks represented by $r^{(2,m)}$ are placed in the lower part of the board, then $\omega_{x,\mathcal{B},\mathcal{A},(m)}(r^{(2,m)}) = -1$. Thus, the total weight over all placements in the second column is of $\mathcal{B}_x^{\mathcal{A},(m)}$ is $(x^m + (b_2)^m + (A_2)^m - (A_2)^m) = (x^m + (b_2)^m)$.

Now suppose that in the first $j-1$ columns of our board, we have placed rooks in s columns of the upper part of the board. It follows that the cells in the a_i-th part of the of the upper and lower augmented boards are cancelled in the j column for $i = j-s+1, \ldots, j$. Then, if we go to place rooks in the j^{th} column of $\mathcal{B}_x^{\mathcal{A},(m)}$, there are x^m ways of placing these rooks in the x-part of the board, $(b_j)^m$ ways of placing rooks in the $\mathcal{B}$-part of the board, $(A_{j-s})^m$ ways of placing rooks in the upper augmented part of the board, and $(A_{j-s})^m$ ways of placing rooks in the lower augmented part of the board, but these placements come with an overall weighting of $(x^m + (b_j)^m + (A_{j-s})^m - (A_{j-s})^m) = (x^m + (b_j)^m)$. Thus,

$$S(\mathcal{B}_x^{\mathcal{A},(m)}) = \prod_{i=1}^{n} (x^m + (b_i)^m),$$

which is the lefthand side of 4.10.

Now suppose that we fix a rook placement $\mathbb{P} \in \mathcal{N}_{n-k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)})$. We now wish to compute

$$\sum_{\substack{Q \in \mathcal{N}_{n,(m)}(\mathcal{B}_x^{\mathcal{A},(m)}) \\ Q \cap \mathcal{B}^{\mathcal{A},(m)} = \mathbb{P}}} W_{x,\mathcal{B},\mathcal{A},(m)}(Q).$$

Each Q will arise from placing m rooks below the high bar in each column which does not contain a rook of $\mathbb{P}$. There are k such columns, and we will begin placing the rooks in these columns by starting with the leftmost available column and working to the right. We can show by essentially the same proof that we used to prove Lemma 2.3, that the number of uncancelled cells in the lower augmented part of the board in the these columns are $A_1, \ldots, A_k$, reading from left to right. Thus in the first available column, there will be x^m ways to place rooks in the x-part of this column and $(A_1)^m$ ways of placing rooks in this column in the lower augmented part. The total weighting for those placements will be $(x^m - (A_1)^m)$. Now, suppose that we are placing a rook in the i^{th} available column, reading from left to right. Then there will be x^m ways of placing m rooks in the x-part of this this column and below the low bar in this column, there will be $(A_i)^m$ ways of placing the m rooks. So, the total weighting for this column will be $(x^m - (A_i)^m)$. We can now see that

$$\sum_{\substack{Q \in \mathcal{N}_{n,(m)}(\mathcal{B}_x^{\mathcal{A},(m)}) \\ Q \cap \mathcal{B}^{\mathcal{A},(m)} = \mathbb{P}}} W_{x,\mathcal{B},\mathcal{A},(m)}(Q) = W_{\mathcal{B},\mathcal{A},(m)}(\mathbb{P})(x^m - (A_1)^m) \cdots (x^m - (A_k)^m).$$

Thus,

$$\begin{aligned} S(\mathcal{B}_x^{\mathcal{A},(m)}) &= \textstyle\sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k,(m)}(\mathcal{B}^{\mathcal{A},(m)})} (x^m - (A_1)^m) \cdots (x^m - (A_k)^m) \\ &= \textstyle\sum_{k=0}^{n} r_{n-k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)}) x^m (x^m - (A_1)^m) \cdots (x^m - (A_k)^m). \end{aligned}$$

□

We note that Theorem 4.3 is just a slightly modified version of the general product formula theorems from Chapter 2, with $sgn(i) = 1$ and $\overline{sgn}(i) = -1$ for all i. For this theorem, we could choose $\mathcal{B}$, $\mathcal{A}$, sgn, and $\overline{sgn}$ so as to get a variety of product formulas, including the following theorem, which is completely analogous to Theorem 1.23.

Corollary 4.4. *Given* $x, n \in \mathbb{N}$ *and* $m \in \mathbb{Z}^+$, *let* $\mathcal{B} = (|b_1|, |b_2 - 1|, \ldots, |b_n - (n-1)|)$ *and* $\mathcal{A} = (0, 1, 1, \ldots, 1)$ *be two sequences in* $\mathbb{N}^n$. *If* $\overline{sgn}(i) = -1$ *for every* i, *then for* *the appropriate choice of* sgn,

$$\prod_{i=1}^{n}(x^m+(b_i-(i-1))^m) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k,(m)}(\mathcal{B}^{\mathcal{A},(m)}, sgn)x^m(x^m-1^m)\cdots(x^m-(k-1)^m). \tag{4.11}$$

Proof Sketch: We need only define $sgn(i) = 1$ when $b_i - (i-1) \geq 0$ and let $sgn(i) = -1$ otherwise. □

4.3 x^m-Stirling Numbers of the First & Second Kind

Consider the poly-Stirling numbers of the first and second kind defined by the the recursions given by Equations (4.2), (4.1), and (4.3). In the special case where $p(x) := x^m$ for some positive integer m, then we call these numbers $S^{x^m}_{n,k}$, $s^{x^m}_{n,k}$, and $c^{x^m}_{n,k}$ the *x^m-Stirling numbers*. In this section, we shall discuss these numbers and their connections with m-rook numbers and m-file numbers.

The *x^m-Stirling numbers of the first kind*, $s^{x^m}_{n,k}$, are defined by the following recursions:

$$s^{x^m}_{0,0} = 1 \text{ and } s^{x^m}_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n \ \ and \tag{4.12}$$

$$s^{x^m}_{n+1,k} = s^{x^m}_{n,k-1} - n^m s^{x^m}_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

We now define $s^{x^m}_{n,k} = (-1)^{n-k}c^{x^m}_{n,k}$. Thus, the integers $c^{x^m}_{n,k}$, called the *signless x^m-Stirling numbers of the first kind*, satisfy the recursion:

$$c^{x^m}_{0,0} = 1 \text{ and } c^{x^m}_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n \ \ and \tag{4.13}$$

$$c^{x^m}_{n+1,k} = c^{x^m}_{n,k-1} + n^m c^{x^m}_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

Finally, we will denote the *x^m-Stirling numbers of the second kind* [1] by $S^{x^m}_{n,k}$, and we will define them by the recursion:

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[1] In the case where $m = 2$, these numbers are discussed in both [25] and [27] where they are referred to as *triangle central factorial numbers.*

$$\begin{aligned} &S^{x^m}_{0,0} = 1 \text{ and } S^{x^m}_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n \;\; and \\ &S^{x^m}_{n+1,k} = S^{x^m}_{n,k-1} + k^m S^{x^m}_{n,k} \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.14}$$

It is a direct result of Milne Inversion that the x^m-Stirling numbers of the first and second kind are inverses of each other, although we will give a combinatorial proof of this fact in the next section. Moreover, notice that when $m = 1$, then the x^m-Stirling numbers defined here match exactly with those defined in Chapter 1.

4.3.1 Rook Theoretic Interpretations of x^m-Stirling Numbers

Theorem 4.5. *If $\boldsymbol{B}_n$ is the staircase board with n columns, then $c^{x^m}_{n,k} = f_{n-k,(m)}(\boldsymbol{B}^{(m)}_n)$.*

Proof: If $n, k < 0$ or $k > n$, then $f_{n-k,(m)}(\mathbf{B}^{(m)}_n) = 0$, and if $n = 0$ we will say that $f_{0,(m)}(\emptyset) = 0$. Thus, the initial conditions have been satisfied. Suppose $n > 0$.

We will now proceed by induction on the number of columns of the board $\mathbf{B}_n$. When $n = 1$ we have $f_{1,(m)}(\mathbf{B}^{(m)}_1) = 0 = c^{x^m}_{1,0}$ and $f_{0,(m)}(\mathbf{B}^{(m)}_1) = 1 = c^{x^m}_{1,1}$. So assume that $n > 1$ and $f_{n-k,(m)}(\mathbf{B}^{(m)}_n) = c^{x^m}_{n,k}$ for $0 \leq k \leq n$. By our definition, $f_{n+1-k,(m)}(\mathbf{B}^{(m)}_{n+1})$ will be equal to the number of rook file placements into the $n+1-k$ columns of $\mathbf{B}^{(m)}_{n+1}$. These placements will either have no rooks placed in the last column of $\mathbf{B}^{(m)}_{n+1}$ or m rooks placed in the last column. Those placements which have no rooks in the last column of $\mathbf{B}^{(m)}_{n+1}$ are counted by $f_{n+1-k,(m)}(\mathbf{B}^{(m)}_n)$, which is equal to $c^{x^m}_{n,k-1}$ by our induction hypothesis. Similarly, if there are rooks placed in the last column of $\mathbf{B}^{(m)}_{n+1}$, then there are only $n-k$ columns of the first n which contain rooks, and we have assumed that those are counted by $f_{n-k,(m)}(\mathbf{B}^{(m)}_n) = c^{x^m}_{n,k}$. Now we need to extend these placements in $\mathbf{B}^{(m)}_n$ to placements in the board $\mathbf{B}^{(m)}_{n+1}$. Since none of the rooks in the first n column of $\mathbf{B}^{(m)}_{n+1}$, cancel to their right, there will be n^m ways to place m rooks in rightmost column of the board. Thus,

$$\begin{aligned} f_{n+1-k,(m)}(\mathbf{B}_{n+1}^{(m)}) &= f_{n+1-k,(m)}(\mathbf{B}_n^{(m)}) + n^m f_{n-k,(m)}(\mathbf{B}_n^{(m)}) \\ &= c_{n,k-1}^{x^m} + n^m c_{n,k}^{x^m} \\ &= c_{n+1,k}^{x^m}. \end{aligned}$$ □

The special case of Theorem 4.2 where $B = \mathbf{B}_n$ gives the following corollary.

Corollary 4.6. *Suppose* $x, n \in \mathbb{N}$. *Then, for any positive integer* m,

$$\prod_{i=1}^{n}(x^m + (i-1)^m) = \sum_{k=0}^{n} c_{n,k}^{x^m}(x^m)^k. \tag{4.15}$$

Corollary 4.7. *Suppose* $x, n \in \mathbb{N}$. *Then, for any positive integer* m,

$$\prod_{i=1}^{n}(x^m - (i-1)^m) = \sum_{k=0}^{n} s_{n,k}^{x^m}(x^m)^k. \tag{4.16}$$

Proof: Consider Equation (4.15). If we replace x^m with $-x^m$ and multiply both sides by $(-1)^n$, then we obtain Equation (4.16). □

Theorem 4.8. *If* $\boldsymbol{B}_n$ *is the staircase board with n columns, then* $S_{n,k}^{x^m} = r_{n-k,(m)}(\boldsymbol{B}_n^{(m)})$.

Proof: If $n, k < 0$ or $k > n$, then $r_{n-k,(m)}(\mathbf{B}_n^{(m)}) = 0$, and if $n = 0$ we will say that $r_{0,(m)}(\emptyset) = 0$. Thus, the initial conditions have been satisfied. Suppose $n > 0$.

We will now proceed by induction on the number of columns of the board $\mathbf{B}_n$. When $n = 1$ we have $r_{1,(m)}(\mathbf{B}_1^{(m)}) = 0 = S_{1,0}^{x^m}$ and $r_{0,(m)}(\mathbf{B}_1^{(m)}) = 1 = S_{1,1}^{x^m}$. So assume that $n > 1$ and $r_{n-k,(m)}(\mathbf{B}_n^{(m)}) = S_{n,k}^{x^m}$ for $0 \leq k \leq n$. By our definition, $r_{n+1-k,(m)}(\mathbf{B}_{n+1})$ will be equal to the number of placements of attacking rooks into $n+1-k$ columns of $\mathbf{B}_{n+1}^{(m)}$. Following the proof of Theorem 4.5, these placements will either have no rooks placed in the last column of $\mathbf{B}_{n+1}^{(m)}$ or m rooks placed in the last column. Those placements which have no rooks in the last column of $\mathbf{B}_{n+1}^{(m)}$ are counted by $r_{n+1-k,(m)}(\mathbf{B}_n^{(m)})$, which is equal to $S_{n,k-1}^{x^m}$ by our induction hypothesis. Similarly, if there are rooks placed in the last column of $\mathbf{B}_{n+1}^{(m)}$, then there are only $n-k$ columns of the first n which contain rooks, and we have assumed that those are counted by $r_{n-k,(m)}(\mathbf{B}_n^{(m)}) = S_{n,k}^{x^m}$. Now we need to extend these placements in $\mathbf{B}_n^{(m)}$ to placements in the board $\mathbf{B}_{n+1}^{(m)}$, and there will be k^m ways to do that. Thus we have

$$\begin{aligned} r_{n+1-k,(m)}(\mathbf{B}_{n+1}^{(m)}) &= r_{n+1-k,(m)}(\mathbf{B}_n^{(m)}) + k^m r_{n-k,(m)}(\mathbf{B}_n^{(m)}) \\ &= S_{n,k-1}^{x^m} + k^m S_{n,k}^{x^m} \\ &= S_{n+1,k}^{x^m}. \end{aligned}$$

□

Lemma 4.9. *Let* $\mathcal{B} = (0,0,\ldots,0)$ *and* $\mathcal{A} = (0,1,\ldots,1)$ *be sequences in* $\mathbb{N}^n$ *and suppose* m *is a positive integer. Then for every* $0 \le k \le n$,

$$r_{k,(m)}(\boldsymbol{B}_n^{(m)}) = r_{k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)}).$$

Proof Sketch: We can check, as we have done with similar theorems, that the numbers $r_{k,(m)}(\mathbf{B}_n^{(m)})$ satisfy the same recursions and initial conditions as the numbers $r_{k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)})$ for the given $\mathcal{B}$ and $\mathcal{A}$. □

This yields the following corollary to Theorem 4.3.

Corollary 4.10. *Suppose* $x, n \in \mathbb{N}$. *Then, for any positive integer* m,

$$(x^m)^n = \sum_{k=0}^{n} S_{n,k}^{x^m}(x^m)(x^m-1^m)\cdots(x^m-(k-1)^m). \tag{4.17}$$

We will give a computational proof of this corollary later in this section, which will be based on the following theorem.

Theorem 4.11. *The lower triangular matrices defined by* $||S^{x^m}_{n,k}||$ *and* $||s^{x^m}_{n,k}||$ *are inverses of one another.*

Proof: (This theorem is an immediate consequence of Milne Inversion Theorem, however, here we will provide a combinatorial proof.)

Consider the sum

$$S(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} S^{x^m}_{n,k} s^{x^m}_{k,j}. \tag{4.18}$$

By definition, $s^{x^m}_{n,k} = (-1)^{n-k} c^{x^m}_{n,k}$, so we have

$$S(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} S^{x^m}_{n,k} c^{x^m}_{k,j}. \tag{4.19}$$

Now, we can think of this sum as representing a weighting over pairs of rook placements $(U,V) \in (\mathcal{N}_{n-k,(m)}(\mathbf{B}^{(m)}_n), \mathcal{F}_{k-j,(m)}(\mathbf{B}^{(m)}_k))$. That is, if for any Ferrers board B we define $w(U) = (1)^k = 1$ for every $U \in \mathcal{N}_{k,(m)}(B)$ and $\tilde{w}(V) = (-1)^k$ for every $V \in \mathcal{F}_{k,(m)}(B)$, then Equation 4.19 becomes

$$S(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{(U,V) \in (\mathcal{N}_{n-k,(m)}(\mathbf{B}^{(m)}_n), \mathcal{F}_{k-j,(m)}(\mathbf{B}^{(m)}_k))} w(U)\tilde{w}(V). \tag{4.20}$$

We now consider the involution I with the following properties:

1. If for $(U,V) \in (\mathcal{N}_{n-k,(m)}(\mathbf{B}^{(m)}_n), \mathcal{F}_{k-j,(m)}(\mathbf{B}^{(m)}_k))$, U has a m rooks in its last column, then $I(U,V) = (U^*,V^*) \in (\mathcal{N}_{n-k-1,(m)}(\mathbf{B}^{(m)}_n), \mathcal{F}_{k-j,(m)}(\mathbf{B}^{(m)}_{k+1}))$, and
   (a) U^* is the placement U with the rooks in the last column removed, and
   (b) if U had a rook in $C_{(l,n)}$ in the w^{th} available cell from the bottom of the board $\mathbf{B}^{(m)}_n$, then V^* is the placement V copied into the larger board, $\mathbf{B}^{(m)}_{k+1}$, with a rook placed in the cell $c(w,l,k)$ of $\mathbf{B}^{(m)}_{k+1}$.

2. If for $(U^*, V^*) \in (\mathcal{N}_{n-k-1,(m)}(\mathbf{B}_n^{(m)}), \mathcal{F}_{k-j+1,(m)}(\mathbf{B}_{k+1}^{(m)}))$, U^* has no rooks in its last column but V^* does, then we reverse the above step.

   An example of this part of the involution can be seen Figure 4.9, where (U, V) and (U^*, V^*) are shown. Here $(U, V) \in (\mathcal{N}_{2,(3)}(\mathbf{B}_5^{(3)}), \mathcal{F}_{2,(3)}(\mathbf{B}_3^{(3)}))$, and $(U^*, V^*) \in (\mathcal{N}_{1,(3)}(\mathbf{B}_5^{(3)}), \mathcal{F}_{3,(3)}(\mathbf{B}_4^{(3)}))$.

3. If for $(U, V) \in (\mathcal{N}_{n-k,(m)}(\mathbf{B}_n^{(m)}), \mathcal{F}_{k-j,(m)}(\mathbf{B}_k^{(m)}))$ neither U nor V has m rooks in the last column, then remove the minimum number of columns, s, from both boards such that at least one of the two placements remaining now has m rooks in the last column. We now have a new pair $(\hat{U}, \hat{V}) \in (\mathcal{N}_{n-k,(m)}(\mathbf{B}_{n-s}^{(m)}), \mathcal{F}_{k-j,(m)}(\mathbf{B}_{k-s}^{(m)}))$. We now repeat the above steps of I on $(\hat{U}, \hat{V})$ to get a pair $(\hat{U}^*, \hat{V}^*)$.

An example of this part of the involution can be seen Figure 4.10, where (U, V), $(\hat{U}, \hat{V})$, and $(\hat{U}^*, \hat{V}^*)$ are shown. Here $(U, V) \in (\mathcal{N}_{2,(3)}(\mathbf{B}_5^{(3)}), \mathcal{F}_{2,(3)}(\mathbf{B}_3^{(3)}))$ with neither containing a rook in the last column, but both containing rooks in the second column from the right, and thus $s = 1$. Once we remove the last column of each board, we get new placements $(\hat{U}, \hat{V})$ and from there, since $\hat{V}$ contains a rooks in its last column, we can recursively define I to give us $(\hat{U}^*, \hat{V}^*)$.

Now, it is clear from I's definition that $I(I(U, V)) = (U, V)$. Moreover, if $w(U)\tilde{w}(V) = +1$, then $w(U^*)\tilde{w}(V^*) = -1$ and also if $w(U)\tilde{w}(V) = -1$, then $w(U^*)\tilde{w}(V^*) = +1$, thus I is a sign-reversing involution. We can now see that, through I, that unless $I(U, V) = (U, V)$ each pair of placements will have a counterpart (U^*, V^*) such that $w(U)\tilde{w}(V) + w(U)\tilde{w}(V^*) = 0$. Thus,

$$S(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{\substack{(U,V) \in (\mathcal{N}_{n-k,(m)}(\mathbf{B}_n^{(m)}), \mathcal{F}_{k-j,(m)}(\mathbf{B}_k^{(m)})) \\ I(U,V)=(U,V)}} w(U)\tilde{w}(V).$$

However, the only fixed points of I are those placement pairs which have no rooks in either placement. That is, for fixed n, k must equal n and j must equal k, or equivalently, $w(U)\tilde{w}(V) = (1)(-1)^{k-j} = 1 = \chi(n = j)$. □

Figure 4.9: An example of the involution I, where either U or V contains a rooks in the last column. Here, $n = 5$, $j = 1$, and k goes between 3 (top) and 4 (bottom).

Figure 4.10: An example of the involution I, where neither U nor V contains a rooks in the last column. Here ,we remove the last column of those boards which contain U and V to get smaller boards and the placements $(\hat{U}, \hat{V})$. From there, we reapply I to get $(\hat{U}^*, \hat{V}^*)$.

As the following proof shows, Corollary 4.10 can also be thought of as a corollary to Theorem 4.11 and Corollary 4.7.

Computational Proof of Corollary 4.10:

$$\begin{aligned}
\textstyle\sum_{k=0}^{n} S_{n,k}^{x^m} \prod_{j=1}^{k}(x^m - (j-1)^m) &= \textstyle\sum_{k=0}^{n} S_{n,k}^{x^m} \sum_{j=1}^{k} s_{k,j}^{x^m} (x^m)^j \\
&= \textstyle\sum_{k=0}^{n} \sum_{j=1}^{k} S_{n,k}^{x^m} s_{k,j}^{x^m} (x^m)^j \\
&= \textstyle\sum_{j=0}^{n} \chi(n = j)(x^m)^j \quad \text{by Thm. 4.11} \\
&= (x^m)^n.
\end{aligned}$$

□

4.3.2 Set and Cycle Structure Interpretations of x^m-Stirling Numbers

In Chapter 1 we saw that Stirling numbers are intimately related to set partitions and cycle structures. As we will see in this section, x^m-Stirling numbers have combinatorial interpretations relating to m-tuples of set partitions and cycles.

Let $P_{n,k}^{(m)} = \{P_1, P_2, \dots P_m\}$ be an m-tuple of unordered set partitions of $[n]$ into k parts. Define $\Pi_{n,k}^{(m)} := \{P_{n,k}^{(m)} |$ the parts of P_i and P_j have the same minimal elements for every $1 \le i < j \le m\}$.

Theorem 4.12. *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. For every $0 \le k \le n$, $S_{n,k}^{x^m} = |\Pi_{n,k}^{(m)}|$.*

Proof: Fix $m \in \mathbb{Z}^+$. First we note that $|\Pi_{n,k}^{(m)}| = 0$ whenever $n < 0$, $k < 0$, or $n < k$, and $\Pi_{0,0}^{(m)} = \{\emptyset\}$, so $|\Pi_{0,0}^{(m)}| = S_{0,0}^{x^m} = 1$. For $n = 1$, $|\Pi_{1,0}^{(m)}| = S_{1,0}^{x^m} = 0$ and $|\Pi_{1,1}^{(m)}| = S_{1,1}^{x^m} = 1$ since $\Pi_{1,1}^{(m)} = \{\{1\}^m\}$. Proceeding by induction, we will pick $n > 1$ and assume that $|\Pi_{n,k}^{(m)}| = S_{n,k}^{x^m}$ for every $0 \leq k \leq n$.

Suppose that $P_{n+1,k}^{(m)} \in \Pi_{n+1,k}^{(m)}$. If $\{n+1\}$ is in a part by itself in $P_i \in P_{n+1,k}^{(m)}$, then $\{n+1\}$ is in a part by itself in $P_j \in P_{n+1,k}^{(m)}$ for every $j = 1, 2, \ldots, m$. Thus we can transform an m-tuple $P_{n,k-1}^{(m)} \mapsto P_{n+1,k}^{(m)}$ by adding the part $\{n+1\}$ to every partition of $P_{n,k-1}^{(m)}$. Similarly, if $\{n+1\}$ is not in a part by itself for some $P_i \in P_{n+1,k}^{(m)}$, then $\{n+1\}$ is not in a part by itself in any $P_j \in P_{n+1,k}^{(m)}$ for $j = 1, 2, \ldots, m$. Thus, we can transform an m-tuple $P_{n,k}^{(m)} \mapsto P_{n+1,k}^{(m)}$ by adding $n+1$ to any of the k parts of each partition in $P_{n,k}^{(m)}$, of which there are k^m ways of doing this.

Thus,

$$\begin{aligned} |\Pi_{n+1,k}^{(m)}| &= |\Pi_{n,k-1}^{(m)}| + k^m |\Pi_{n,k}^{(m)}| \\ &= S_{n,k-1}^{x^m} + k^m S_{n,k}^{x^m} \\ &= S_{n+1,k}^{x^m}. \end{aligned}$$

□

Theorem 4.13. *Let* $C_{n,k}^{(m)} = \{C_1, C_2, \ldots, C_m\}$ *be an* m*-tuple of permutations of* $[n]$ *with k cycles. If we define* $\Omega_{n,k}^{(m)} = \{C_{n,k}^{(m)} \,|$ *the cycles of* C_i *and* C_j *have the same minimal elements for every* $1 \leq i < j \leq m\}$*, then* $c_{n,k}^{x^m} = |\Omega_{n,k}^{(m)}|$ *for every* $0 \leq k \leq n$.

Proof: Fix $m \in \mathbb{Z}^+$. First we note that $|\Omega_{n,k}^{(m)}| = 0$ whenever $n < 0$, $k < 0$, or $n < k$, and $\Omega_{0,0}^{(m)} = \{\emptyset\}$, so $|\Omega_{0,0}^{(m)}| = c_{0,0}^{x^m} = 1$. For $n = 1$, $|\Omega_{1,0}^{(m)}| = c_{1,0}^{x^m} = 0$ and $|\Omega_{1,1}^{(m)}| = c_{1,1}^{x^m} = 1$ since $\Omega_{1,1}^{(m)} = \{(1)^m\}$. Proceeding by induction, we will pick $n > 1$ and assume that $|\Omega_{n,k}^{(m)}| = c_{n,k}^{x^m}$ for every $0 \leq k \leq n$.

Suppose that $C_{n+1,k}^{(m)} \in \Omega_{n_1,k}^{(m)}$. If $(n+1)$ is a cycle in $C_i \in C_{n+1,k}^{(m)}$, then $(n+1)$ is a cycle in $C_j \in C_{n+1,k}^{(m)}$ for every $j = 1, 2, \ldots, m$. Thus we can transform an m-tuple $C_{n,k-1}^{(m)} \mapsto C_{n+1,k}^{(m)}$ by adding the cycle $(n+1)$ to every collection of cycles of $C_{n,k-1}^{(m)}$. Similarly, if $(n+1)$ is not a cycle for some $C_i \in C_{n+1,k}^{(m)}$, then $(n+1)$ is not cycle in any $C_j \in C_{n+1,k}^{(m)}$ for $j = 1, 2, \ldots, m$. Thus, we can transform an

Figure 4.11: Type I q-counting for a non-attacking and file rook placements in the board $B^{(3)}$, with $B = F(1,2,2,4,5)$.

m-tuple $C^{(m)}_{n,k} \mapsto C^{(m)}_{n+1,k}$ by inserting $n+1$ immediately after one of the elements in each of the cycle structures in $C^{(m)}_{n,k}$, of which there are n^m ways of doing this.

Thus,

$$\begin{aligned} |\Omega^{(m)}_{n+1,k}| &= |\Omega^{(m)}_{n,k-1}| + n^m |\Omega^{(m)}_{n,k}| \\ &= c^{x^m}_{n,k-1} + n^m c^{x^m}_{n,k} \\ &= c^{x^m}_{n+1,k}. \end{aligned}$$ □

Another theorem involving these x^m-Stirling numbers is the following, which gives us a generating function for the $S^{x^m}_{n,k}$'s.

Theorem 4.14. *For $k \in \mathbb{N}$,*

$$\sum_{n \geq k} S^{x^m}_{n,k} x^n = \frac{x^k}{(1-1^m x)(1-2^m x)\cdots(1-k^m x)}.$$

4.4 Q-Analogues of x^m-Stirling Numbers

4.4.1 Type I Q-Analogues

To study the q-analogues of *x^m-Stirling numbers*, we must first define how to q-count rook placements in a general Ferrers board $B^{(m)}$. For any placement $\mathbb{P} \in \mathcal{N}_{k,(m)}(B^{(m)})$, let *unc*$_{\mathcal{N}}(\mathbb{P})$ denote the number of uncancelled cells of $\mathbb{P}$ which lie above a rook. For example, on the lefthand side of Figure 4.11, we have a pictured a non-attacking rook placement $\mathbb{P}$ where *unc*$_{\mathcal{N}}(\mathbb{P}) = q^5$. Now, given a board $B^{(m)}$ we will define the k^{th} *type I qm-rook number of* $B^{(m)}$, denoted by $r_{k,(m)}(B^{(m)}, q)$, as follows:

$$r_{k,(m)}(B^{(m)}, q) = \sum_{\mathbb{P} \in \mathcal{N}_{k,(m)}(B^{(m)})} q^{unc_{\mathcal{N}}(\mathbb{P})}. \tag{4.21}$$

Suppose that $B = F(b_1, b_2, \ldots, b_n)$ is a Ferrers board and $m \in \mathbb{Z}^+$. From the definition of $r_{k,(m)}(B^{(m)}, q)$, we can see that the type I qm-rook numbers satisfy the following recursions:

$$\begin{aligned} & r_{0,(m)}(B^{(m)}, q) = 1 \text{ and } r_{k,(m)}(B^{(m)}, q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \\ & r_{n+1-k,(m)}(B^{(m)}, q) = r_{n+1-k,(m)}(B^{(m)}/b_n, q) \\ & \qquad\qquad + \ [b_n - (n-k)]_q^m r_{n-k,(m)}(B^{(m)}/b_n, q) \\ & \quad \text{if } 0 \leq k \leq n+1 \text{ and } n \geq 0 \end{aligned} \tag{4.22}$$

where $B^{(m)}/b_n$ denotes the board which results by removing the n^{th} column from $B^{(m)}$. We wish to define the *type I qx^m-Stirling numbers of the second kind* by the recursions

$$\begin{aligned} & S_{0,0}^{x^m}(q) = 1 \text{ and } S_{n,k}^{x^m}(q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \\ & S_{n+1,k}^{x^m}(q) = S_{n,k-1}^{x^m}(q) + ([k]_q)^m S_{n,k}^{x^m}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.23}$$

Then we can see that in the special case where $B = \mathbf{B}_n$, then the polynomials $S_{n,k}^{x^m}(q)$ and the polynomials $r_{n-k}(\mathbf{B}_n^{(m)}, q)$ satisfy the same recursions with identical initial conditions.

We can also define, for any placement $\mathbb{P} \in \mathcal{F}_{k,(m)}(B^{(m)})$, the k^{th} *Type I qm-file number of $B^{(m)}$*, denoted by $f_{k,(m)}(B^{(m)}, q)$, to be

$$f_{k,(m)}(B^{(m)}, q) = \sum_{\mathbb{P} \in \mathcal{F}_{k,(m)}(B^{(m)})} q^{unc_{\mathcal{F}}(\mathbb{P})}, \tag{4.24}$$

where $unc_{\mathcal{F}}(\mathbb{P})$ is equal to the number of cells in $\mathbb{P}$ which lie directly above a rook in $\mathbb{P}$. If we look at the example in righthand side of Figure 4.11, we see that particular placement would give a q-count of $unc_{\mathcal{F}}(\mathbb{P}) = q^6$.

Suppose we are given any rook board $B = F(b_1, b_2, \ldots, b_n)$ and $m \in \mathbb{Z}^+$. From the definition of $f_{k,(m)}(B^{(m)}, q)$, we can see that the type I qm-file numbers satisfy the following recursions:

$$\begin{aligned} &f_{0,(m)}(B^{(m)},q)=1 \text{ and } f_{k,(m)}(B^{(m)},q)=0 \text{ if } k<0 \text{ or } k>n, \text{ and} \\ &f_{n+1-k,(m)}(B^{(m)},q)=f_{n+1-k,(m)}(B^{(m)}/b_n,q)+[b_n]_q^m f_{n-k,(m)}(B^{(m)}/b_n,q) \\ &\qquad \text{if } 0\le k\le n+1 \text{ and } n\ge 0. \end{aligned} \tag{4.25}$$

We wish to define the *signless type I qx^m-Stirling numbers of the first kind* by the recursions

$$\begin{aligned} &c^{x^m}_{0,0}(q)=1 \text{ and } c^{x^m}_{n,k}(q)=0 \text{ if } k<0 \text{ or } k>n, \text{ and} \\ &c^{x^m}_{n+1,k}(q)=c^{x^m}_{n,k-1}(q)+([n]_q)^m c^{x^m}_{n,k}(q) \text{ if } 0\le k\le n+1 \text{ and } n\ge 0, \end{aligned} \tag{4.26}$$

and we can see that in the special case of $B=\mathbf{B}_n$, that these satisfy the same recursions as $f_{n-k,(m)}(\mathbf{B}_n^{(m)},q)$. Also, by replacing $c^{x^m}_{n,k}(q)$ with $(-1)^{n-k}s^{x^m}_{n,k}(q)$, we get the following recursions:

$$\begin{aligned} &s^{x^m}_{0,0}(q)=1 \text{ and } s^{x^m}_{n,k}(q)=0 \text{ if } k<0 \text{ or } k>n, \text{ and} \\ &s^{x^m}_{n+1,k}(q)=s^{x^m}_{n,k-1}(q)-([n]_q)^m s^{x^m}_{n,k}(q) \text{ if } 0\le k\le n+1 \text{ and } n\ge 0, \end{aligned} \tag{4.27}$$

where the $s^{x^m}_{n,k}(q)$'s are referred to as the *type I qx^m-Stirling numbers of the first kind*.

The Milne Inversion Theorem 4.1 then implies the following corollary.

Corollary 4.15. *The matrices $||S^{x^m}_{n,k}(q)||$ and $||s^{x^m}_{n,k}(q)||$ are inverses of each other.*

Now consider the $\mathbb{P}\in\mathcal{F}_{n,(m)}(B_x^{(m)})$ and define

$$\Delta_{x,B^{(m)}}(\mathbb{P},q)=\prod_{j=1}^{n}\Delta_{C,x,B^{(m)}}(C_j,q).$$

Then set

$$\Delta_{C,x,B^{(m)}}(C_j, q) = \prod_{l=1}^{m} \delta_{x,B^{(m)}}(r_{(l,j)}, q),$$

where

1. if $r_{(l,j)}$ is in cell $c(i,l,j)$, then $\delta_{x,B^{(m)}}(r_{(l,j)}, q) = q^{|\{c(s,l,j):i<s\leq b_j\}|}$, and
2. if $r_{(l,j)}$ is in cell $c_x(i,l,j)$, then $\delta_{x,B^{(m)}}(r_{(l,j)}, q) = q^{|\{c_x(s,l,j):1\leq s<i\}|}$.

Theorem 4.16. *Let* $x, n \in \mathbb{N}$ *and* $m \in \mathbb{Z}^+$. *If* $B = F(b_1, b_2, \ldots, b_n)$ *is any rook board, then*

$$\prod_{i=1}^{n}([x]_q^m + [b_i]_q^m) = \sum_{k=0}^{n} f_{n-k,(m)}(B^{(m)}, q)([x]_q^m)^k. \tag{4.28}$$

Proof: Let

$$S(B_x^{(m)}, q) = \sum_{Q\in\mathcal{F}_{n,(m)}(B_x^{(m)})} \Delta_{x,B^{(m)}}(Q, q).$$

We then claim that Equation (4.28) represents computing $S(B_x^{(m)}, q)$ in two different ways.

In the first way, we will place m rooks in each column of $B_x^{(m)}$, starting with the leftmost column and working to the right. In the first column, if we place the m rooks in the x-part of the board, then there are x^m way to do this, and we will get a total q-weight, over all of these x^m possibilities, of $[x]_q^m$. Now if we place the m rooks above the high bar, then there are b_1^m ways to place these rooks, and the total q-weight over these b_1^m placements will be $[b_1]_q^m$. Thus, if we consider the $x^m + b_1^m$ possible placements of m rooks in C_1, we get a total q-weight of $[x]_q^m + [b_1]_q^m$ for these placements. In general, the rooks placed in $B_x^{(m)}$ do not cancel to their right, so by the same logic as before, for any $1 \leq j \leq n$, in C_j there will be $x^m + b_j^m$ possible rook placements in that column and they will come with a total q-weight of $[x]_q^m + [b_j]_q^m$. Thus

$$S(B_x^{(m)}, q) = \prod_{i=1}^{n}([x]_q^m + [b_i]_q^m).$$

In the second ways of counting S, suppose we first fix a placement $\mathbb{P} \in \mathcal{F}_{n-k,(m)}(B^{(m)})$, and we extend this to a placement $Q \in \mathcal{F}_{n,(m)}(B_x^{(m)})$. Suppose too that the column of $B_x^{(m)}$ which contain a rook from $\mathbb{P}$ are $C_{i_1}, \ldots, C_{i_{n-k}}$ and the columns which do not contain a rook of $\mathbb{P}$ are $\tilde{C}_{i_1}, \ldots, \tilde{C}_{i_k}$. One can see that each such Q arises by placing m rooks in the x-part of each column of $B_x^{(m)}$ which does not contain a rook from $\mathbb{P}$. In each of these k columns, there will be x^m ways of placing the rooks, and each will contribute a q-weight of $[x]_q^m$. Now, by how we defined the q-weight of $\delta_{x,B^{(m)}}(r_{(l,j)}, q)$,

$$\begin{aligned}\Delta_{x,B^{(m)}}(Q,q) &= (\textstyle\prod_{j=1}^{n-k} \Delta_{C,x,B^{(m)}}(C_{i_j}, q))(\prod_{j=1}^{k} \Delta_{C,x,B^{(m)}}(\tilde{C}_{i_j}, q)) \\ &= q^{unc_{\mathcal{F}}(\mathbb{P})}([x]_q^m)^k.\end{aligned}$$

Summing over all possible k and $\mathbb{P}$, we get that

$$S(B_x^{(m)}, q) = \sum_{k=0}^{n} f_{n-k,(m)}(B^{(m)}, q)([x]_q^m)^k,$$

which is the desired result. □

In the special case where $B = \mathbf{B}_n$, Theorem 4.16 implies the following product formulas involving the $c_{n,k}^{x^m}(q)$'s and the $s_{n,k}^{x^m}(q)$'s:

$$\prod_{i=1}^{n}([x]_q^m + [(i-1)]_q^m) = \sum_{k=0}^{n} c_{n,k}^{x^m}(q)([x]_q^m)^k \tag{4.29}$$

and

$$\prod_{i=1}^{n}([x]_q^m - [(i-1)]_q^m) = \sum_{k=0}^{n} s_{n,k}^{x^m}(q)([x]_q^m)^k. \tag{4.30}$$

From Corollary 4.15, we know that we may also obtain the following product formula involving the $S^{x^m}_{n,k}(q)$'s:

$$([x]^m_q)^n = \sum_{k=0}^{n} S^{x^m}_{n,k}(q) \prod_{j=1}^{k}([x]^m_q - [(j-1)]^m_q). \tag{4.31}$$

However, if we would like a more direct combinatorial interpretation of Equation (4.31), then we will need the following q-analogues of type $\mathcal{A}$ rook numbers.

Consider the n-length sequences of natural numbers $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$. Define for any placement $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})$, the q-weight of $\mathbb{P}$ to be

$$\Omega_{\mathcal{B}^{\mathcal{A},(m)}}(\mathbb{P}, q) = \prod_{j=1}^{k} \Omega_{C,\mathcal{B}^{\mathcal{A},(m)}}(C_{i_j}, q),$$

where C_{i_y} is the y^{th} column from the left of the board which contains m rooks and

$$\Omega_{C,\mathcal{B}^{\mathcal{A},(m)}}(C_{i_j}, q) = \prod_{l=1}^{m} \omega_{\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}),$$

where for each rook $r_{(l,j)} \in \mathbb{P}$,

1. $\omega_{\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}) = |\{c(s,l,j) : i < s \leq b_j\}|$ if $r_{(l,j)}$ is in cell $c(i,l,j)$ of the $\mathcal{B}$-part of $\mathcal{B}^{\mathcal{A},(m)}$, and
2. $\omega_{\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}) = |\{c(s,l,t) : i < s \leq t\}|$ if $r_{(l,j)}$ is in cell $c(i,l,j)$ of the augmented part $\mathcal{B}^{\mathcal{A},(m)}$, where $c(t,l,j)$ is the highest uncancelled cell in $C_{(l,j)}$.

We then define the k^{th} *type $\mathcal{A}$ q-rook number of $\mathcal{B}^{\mathcal{A},(m)}$* to be

$$r^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)}, q) = \sum_{\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k,(m)}(\mathcal{B}^{\mathcal{A},(m)})} \Omega_{\mathcal{B}^{\mathcal{A},(m)}}(\mathbb{P}, q) \tag{4.32}$$

Now define for any placement $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}^{\mathcal{A},(m)}_x)$ the quantity

$$\Omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(\mathbb{P}, q) = \prod_{j=1}^{n} \Omega_{x,C,\mathcal{B}^{\mathcal{A},(m)}}(C_j, q),$$

where C_j is the j^{th} column of the board and

$$\Omega_{x,C,\mathcal{B}^{\mathcal{A},(m)}}(C_j, q) = \prod_{l=1}^{m} \omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q).$$

Now, for the board $\mathcal{B}_x^{\mathcal{A},(m)}$, $\omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q)$ is the q-weight of the rook in column $C_{(l,j)}$ defined by the following:

1. If $r_{(l,j)}$ is in cell $c(i,l,j)$ in the $\mathcal{B}$-part, then $\omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q) = q^{|\{c(s,l,j):i<s\leq b_j\}|}$
2. If $r_{(l,j)}$ is in cell $c_x(i,l,j)$ in the x-part, then $\omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q) = q^{|\{c_x(s,l,j):1\leq s<i\}|}$
3. If $r_{(l,j)}$ is in cell $\tilde{c}(i,l,j)$ in the lower augmented part, then $\omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q) = -q^{|\{\tilde{c}(s,l,j):i<s\leq t\}|}$, where $\tilde{c}(t,l,j)$ is the lowest uncancelled cell in $C_{(l,j)}$.
4. If $r_{(l,j)}$ is in cell $c(i,l,j)$ in the upper augmented part, then $\omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q) = q^{|\{c(s,l,t):i<s\leq t\}|}$, where $c(t,l,j)$ is the highest uncancelled cell in $C_{(l,j)}$.

Now that we have a q-weighting scheme for the board $\mathcal{B}_x^{\mathcal{A},(m)}$, we can prove the following theorem.

Theorem 4.17. *Let $x, n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. If $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$ are sequences in $\mathbb{N}^n$, then,*

$$\prod_{i=1}^{n}([x]_q^m + ([b_i]_q)^m) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k,(m)}(\mathcal{B}^{\mathcal{A},(m)}, q) \prod_{j=1}^{k}([x]_q^m - [A_j]_q^m), \tag{4.33}$$

where $A_i = a_1 + \cdots + a_i$.

Proof: Let

$$W(\mathcal{B}_x^{\mathcal{A},(m)}, q) = \sum_{Q \in \mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}_x^{\mathcal{A},(m)})} \Omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(Q, q).$$

We then claim that Equation (4.33) represents two different ways of computing $W(\mathcal{B}_x^{\mathcal{A},(m)}, q)$.

In the first way, we will place m rooks in each column of $\mathcal{B}_x^{\mathcal{A},(m)}$, starting with the leftmost column and working to the right. In the first column, if we place the m rooks in the x-part of the board, then there are x^m way to do this, and we will get a total q-weight, over all of these x^m possibilities, of $[x]_q^m$. Now if we place the m rooks in the $\mathcal{B}$-part, then there are b_1^m ways to place these rooks, and the total q-weight over these b_1^m placements will be $[b_1]_q^m$. Finally, there are $2(A_1^m)$ ways to place the rooks in the augmented parts of the boards, however, we weight these choices differently. That is, there are A_1^m ways of placing the rooks in the both the upper and lower augmented parts of the board, but the q-weighting over the placements of rooks in the upper augmented part is $[A_1]_q^m$ and the q-weighting over the placements of rooks in the lower augmented part is $-[A_1]_q^m$. Thus, if we consider the $x^m + b_1^m + 2(A_1)^m$ possible placements of m rooks in C_1, we get a total q-weight of $[x]_q^m + [b_1]_q^m + [A_1]_q^m - [A_1]_q^m = [x]_q^m + [b_1]_q^m$ for these placements. When we go to place rooks in the second column of the board, we have two cases.

Case I: Suppose the m rooks in the first column were placed above the high bar. Then, the cells from the a_2-part of the board were cancelled in the second column, and thus, when we go to place m rooks in the second column, we are back in a similar situation as before. That is, we have $x^m + b_2^m + 2(A_1^m)$ ways of placing the rooks. However, there placements give a total q-weight of $[x]_q^m + [b_2]_q^m + [A_1]_q^m - [A_1]_q^m = [x]_q^m + [b_2]_q^m$.

Case II: Suppose that the m rooks placed in the first column were place below the high bar. Then, no cells were cancelled in the second columns, and we will have $x^m + b_2^m + 2(A_2^m)$ ways to place the m rooks. Again though, the total weighting for these placements will be $[x]_q^m + [b_2]_q^m + [A_2]_q^m - [A_2]_q^m = [x]_q^m + [b_2]_q^m$.

In general, suppose we are going to place rooks in the j^{th} column of $\mathcal{B}_x^{\mathcal{A},(m)}$, and suppose that we have placed rooks in s columns above the high bar in the first $j-1$ columns. Then in this column there will be $x^m + b_j^m + 2(A_{j-s}^m)$ ways in which to place the rooks. If we q-count over all of these possible placements however, we see that they will contribute a total q-weight of $[x]_q^m + [b_j]_q^m + [A_{j-s}]_q^m - [A_{j-s}]_q^m = [x]_q^m + [b_j]_q^m$, and so

$$W(\mathcal{B}_x^{\mathcal{A},(m)}, q) = \prod_{i=1}^{n} ([x]_q^m + [b_i]_q^m).$$

In the second way of counting W, suppose we first fix a placement $\mathbb{P} \in \mathcal{N}_{n-k,(m)}(\mathcal{B}^{\mathcal{A},(m)})$.We wish to compute the sum of the weights of all placements $Q \in \mathcal{N}_{n,(m)}(\mathcal{B}_x^{\mathcal{A},(m)})$ such that $Q \cap \mathcal{B}^{\mathcal{A},(m)} = \mathbb{P}$. Suppose that we label the columns of $B_x^{(m)}$ which contain a rook from $\mathbb{P}$ as $C_{i_1}, \ldots, C_{i_{n-k}}$ and the columns which do not contain a rook of $\mathbb{P}$ as $\tilde{C}_{i_1}, \ldots, \tilde{C}_{i_k}$. One can see that each such Q arises by placing m rooks below the high bar in each column of $\mathcal{B}_x^{\mathcal{A},(m)}$ which does not contain a rook from $\mathbb{P}$. In each of these k columns, there will be x^m ways of placing the rooks in the x-part, and each will contribute a q-weight of $[x]_q^m$. There will also be, in column $\tilde{C}_{i_t}$, A_t^m ways of placing rooks in the lower augmented part, but the total q-weight over these placements will be $-[A_t]_q^m$. By how we defined the q-weight of $\omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q)$,

$$\begin{aligned}
\Omega_{x,\mathcal{B}^{\mathcal{A},(m)}}(Q,q) &= \left(\textstyle\prod_{j=1}^{n-k} \Omega_{C,x,\mathcal{B}^{\mathcal{A},(m)}}(C_{i_j}, q)\right)\left(\textstyle\prod_{j=1}^{k} \Omega_{C,x,\mathcal{B}^{\mathcal{A},(m)}}(\tilde{C}_{i_j}, q)\right) \\
&= \Omega_{\mathcal{B}^{\mathcal{A},(m)}}(\mathbb{P}, q) \textstyle\prod_{j=1}^{k}([x]_q^m - [A_j]_q^m).
\end{aligned}$$

Summing over all possible k and $\mathbb{P}$, we get that

$$W(\mathcal{B}_x^{\mathcal{A},(m)}, q) = \sum_{k=0}^{n} r_{n-k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)}, q) \prod_{j=1}^{k}([x]_q^m - [A_j]_q^m),$$

which is the desired result. □

We can now see that Equation (4.31) is also a direct result of this theorem.

4.4.2 Type II Q-Analogues

In this subsection, we will give an alternative way q-count rook and file placement which will lead to different q-analogues of the x^m-Stirling numbers. This alternative way of q-counting rook and file placements is best explained by through an example.

Step 1: Suppose we have a Ferrers board $B = F(1,2,2,4,5)$ and a placement $\mathbb{P} \in \mathcal{N}_{2,(3)}(B^{(3)})$, as in Figure 4.12, where the rooks are placed in columns $C_{i_1}, C_{i_2} = C_2, C_4$.

Figure 4.12: Step 1: A placement of non-attacking rooks in two columns of $B^{(3)}$ with $B = F(1,2,2,4,5)$.

Step 2: We remove all of the rooks of $\mathbb{P}$ from $B^{(m)}$, and we number each subcolumn of $B^{(m)}$, from top to bottom, with the digits $0, 1, 2, \ldots$, as in Figure 4.13.

Figure 4.13: Step 2: A numbering of the blank board $B^{(3)}$ with $B = F(1,2,2,4,5)$.

Step 3: We will place the rooks which were in column $C_{i_1} = C_2$ of the original placement in $B^{(m)}$ back into the numbered board, and cancel in the normal way. We will then note which numbers were in the the cells now filled by these m rooks, and we will assign these rooks a q-weight of $\nu_C(C_{i_1}, q) = q^{(a_1 a_2 \cdots a_m)_{b_{i_1}}}$, where $(a_1 a_2 \cdots a_n)_p$ is the p-ary digit $a_1(p^{n-1}) + a_2(p^{n-2}) + \cdots + a_n(p^0)$. In this case, the rooks placed in column C_{i_1} give us a q-weight of $\nu_C(C_{i_1}, q) = q^{(100)_2} = q^{1(4)+0(2)+0(1)} = q^4$. We will then renumber the remaining uncancelled cells in the columns to the right of C_{i_1} as we did in Step 2. This step can be seen in Figure 4.14.

Figure 4.14: Step 3: We begin to place the original rooks back into the board $B^{(m)}$, and we keep track of the numbers in those cells. We then assign a q-weight to those rooks, and renumber to the right of those rooks.

Step 4: Now we will place the rooks back into column C_{i_2}, which here is the fourth column of $B^{(m)}$. We then assign those rooks a q-weight of $\nu_C(C_{i_2}, q) =$

$q^{(a_1a_2\cdots a_m)_{b_{i_2}}-1}$. Here the rooks in the fourth column of $B^{(m)}$ will be assigned a q-weight of $\nu_C(C_{i_2}, q) = q^{(102)_4-1} = q^{1(9)+0(3)+2(1)} = q^{11}$. This step can be seen in Figure 4.15.

Figure 4.15: Step 4: We repeat Step 3 for the rooks in column C_4.

Step 5: In general, we will, after replacing the rooks in a given column C_{i_w}, give those rooks a q-weight of $\nu_C(C_{i_w}, q) = q^{(a_1a_2\cdots a_m)_{b_{i_w}}-(w-1)}$. We will then define the q-weight of the original placement $\mathbb{P} \in \mathcal{N}_{k,(m)}(B^{(m)})$ to be

$$\nu(\mathbb{P}, q) := \prod_{w=1}^{k} \nu_C(C_{i_w}, q).$$

For this example,

$$\nu(\mathbb{P}, q) = \nu_C(C_{i_1}, q)\nu_C(C_{i_2}, q) = q^4q^{11} = q^{15}.$$

We now define the k^{th} *type II qm-rook number of* $B^{(m)}$ to be

$$\overline{r}_{k,(m)}(B^{(m)}, q) := \sum_{\mathbb{P}\in\mathcal{N}_{k,(m)}(B^{(m)})} \nu(\mathbb{P}, q). \tag{4.34}$$

Let $B = F(b_1, b_2, \ldots, b_n)$ be a Ferrers board and suppose that $m \in \mathbb{Z}^+$. If we again let $B^{(m)}/b_n$ be the board $B^{(m)}$ with the n^{th} column removed, then we can see that these type II q-rook numbers satisfy the following recursions.

$$\begin{aligned} &\overline{r}_{0,(m)}(B^{(m)}, q) = 1 \text{ and } \overline{r}_{k,(m)}(B^{(m)}, q) = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and} \\ &\overline{r}_{n+1-k,(m)}(B^{(m)}, q) = \overline{r}_{n+1-k,(m)}(B^{(m)}/b_n, q) \\ &\qquad\qquad + \ [(b_n - (n-k))^m]_q \overline{r}_{n-k,(m)}(B^{(m)}/b_n, q) \\ &\quad \text{if } 0 \le k \le n+1 \text{ and } n \ge 0. \end{aligned} \tag{4.35}$$

We wish to define the *type II qx^m-Stirling numbers of the second kind* by the recursions:

Figure 4.16: An example of a q-count for a column-strict rook placement $\mathbb{P} \in \mathcal{F}_{k,(m)}(B^{(m)})$, where $B = (0,1,2,4,5,5)$. This placement has a q-weight of q^{120}.

$$\overline{S}^{x^m}_{0,0}(q) = 1 \text{ and } \overline{S}^{x^m}_{n,k}(q) = 0 \text{ if } k < 0 \text{ or } k > n \;\; and \qquad (4.36)$$
$$\overline{S}^{x^m}_{n+1,k}(q) = \overline{S}^{x^m}_{n,k-1}(q) + [k^m]_q \overline{S}^{x^m}_{n,k}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0.$$

Again, we see that in the special case where $B = \mathbf{B}_n$, the polynomials $\overline{S}^{x^m}_{n,k}(q)$ and $\overline{r}_{n-k,(m)}(\mathbf{B}_n^{(m)}, q)$ satisfy the same recursions with identical initial conditions.

There are also type II qm-file numbers, which can be defined in a very similar way to the rook numbers. Here, we define the k^{th} *type II qm-file number of* $B^{(m)}$ to be

$$\overline{f}_{k,(m)}(B^{(m)}, q) := \sum_{\mathbb{P} \in \mathcal{F}_{k,(m)}(B^{(m)})} \mu(\mathbb{P}, q). \qquad (4.37)$$

We define

$$\mu(\mathbb{P}, q) = \prod_{w=1}^{k} \mu_C(C_{i_w}, q)$$

and if the rooks that are placed in C_{i_w} lie in the cells $c(a_1, 1, i_w)$, $c(a_2, 2, i_w)$, …, $c(a_m, m, i_w)$, then

$$\mu_C(C_{i_w}, q) = q^{(a_1 a_2 \cdots a_m)_{b_{i_w}}}.$$

An example of this q-weighting can be seen in Figure 4.16, and the placement shown has a q-weight of

$$\mu(\mathbb{P}, q) = q^{(101)_2} q^{(221)_5} q^{(204)_5} = q^5 q^{61} q^{54} = q^{120}.$$

Given any rook board $B = F(b_1, b_2, \ldots, b_n)$ and any positive integer m, the $\overline{f}_{k,(m)}(B^{(m)}, q)$ satisfy the recursions

$$\begin{aligned}
&\overline{f}_{0,(m)}(B^{(m)},q)=1 \text{ and } \overline{f}_{k,(m)}(B^{(m)},q)=0 \text{ if } k<0 \text{ or } k>n, \text{ and} \qquad (4.38)\\
&\overline{f}_{n+1-k,(m)}(B^{(m)},q)=\overline{f}_{n+1-k,(m)}(B^{(m)}/b_n,q)+[b_n^m]_q\overline{f}_{n-k,(m)}(B^{(m)}/b_n,q)\\
&\qquad \text{if } 0\leq k\leq n+1 \text{ and } n\geq 0.
\end{aligned}$$

We wish to define the *signless type II qx^m-Stirling numbers of the first kind* by the recursions:

$$\begin{aligned}
&\overline{c}^{x^m}_{0,0}(q)=1 \text{ and } \overline{c}^{x^m}_{n,k}(q)=0 \text{ if } k<0 \text{ or } k>n, \text{ and} \qquad (4.39)\\
&\overline{c}^{x^m}_{n+1,k}(q)=\overline{c}^{x^m}_{n,k-1}(q)+[n^m]_q\overline{c}^{x^m}_{n,k}(q) \text{ if } 0\leq k\leq n+1 \text{ and } n\geq 0.
\end{aligned}$$

We see that in the special case of $B=\mathbf{B}_n$ that these satisfy the same recursions as $\overline{f}_{n-k,(m)}(\mathbf{B}_n^{(m)},q)$. Also, by replacing $\overline{c}^{x^m}_{n,k}(q)$ with $(-1)^{n-k}\overline{s}^{x^m}_{n,k}(q)$, we get the following recursions:

$$\begin{aligned}
&\overline{s}^{x^m}_{0,0}(q)=1 \text{ and } \overline{s}^{x^m}_{n,k}(q)=0 \text{ if } k<0 \text{ or } k>n, \text{ and} \qquad (4.40)\\
&\overline{s}^{x^m}_{n+1,k}(q)=\overline{s}^{x^m}_{n,k-1}(q)-[n^m]_q\overline{s}^{x^m}_{n,k}(q) \text{ if } 0\leq k\leq n+1 \text{ and } n\geq 0.
\end{aligned}$$

The $\overline{s}^{x^m}_{n,k}(q)$'s are referred to as the *type II qx^m-Stirling numbers of the first kind.*

Applying the Milne Inversion Theorem 4.1 to these type II q-Stirling numbers, we get the following corollary.

Corollary 4.18. *The matrices $||\overline{S}^{x^m}_{n,k}(q)||$ and $||\overline{s}^{x^m}_{n,k}(q)||$ are inverses of each other.*

Suppose that $\mathbb{P}\in\mathcal{F}_{n,(m)}(B_x^{(m)})$ and define

$$M_{x,B^{(m)}}(\mathbb{P},q)=\prod_{j=1}^{n}M_{C,x,B^{(m)}}(C_j,q),$$

where

Figure 4.17: An example of a q-count for a column-strict rook placement $\mathbb{P} \in \mathcal{F}_{6,(3)}(\mathcal{B}_x^{(3)})$, where $B = (0, 1, 2, 4, 5, 5)$ and $x = 5$. This placement has a q-weight of q^{273}.

1. if the m rooks in C_j lie in the cells $c(a_1, 1, j)$, $c(a_2, 2, j)$, …, $c(a_m, m, j)$, then $M_{C,x,B^{(m)}}(C_j, q) = q^{(a_1 a_2 \cdots a_m)_m}$, and
2. if the m rooks in C_j lie in the cells $c_x(a_1, 1, j)$, $c_x(a_2, 2, j)$, …, $c_x(a_m, m, j)$, then $M_{C,x,B^{(m)}}(C_j, q) = q^{((x-a_1)(x-a_2)\cdots(x-a_m))_m}$.

An example of this type of q-weighting can be seen in Figure 4.17, where the same board and placement as in Figure 4.16 is used above the bar. Here

$$\begin{aligned} M_{5,B^{(3)}}(\mathbb{P}, q) &= \textstyle\prod_{j=1}^{6} M_{C,5,B^{(3)}}(C_j, q) \\ &= q^{(232)_5} q^{(012)_5} q^{(101)_2} q^{(304)_5} q^{(221)_5} q^{(204)_5} \\ &= q^{273}. \end{aligned}$$

Theorem 4.19. *Let $B = F(b_1, b_2, \ldots, b_n)$ be any rook board. If $x, n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$, then*

$$\prod_{i=1}^{n}([x^m]_q + [b_i^m]_q) = \sum_{k=0}^{n} \overline{f}_{n-k,(m)}(B^{(m)}, q)([x^m]_q)^k. \tag{4.41}$$

Proof: Let

$$S(B_x^{(m)}, q) = \sum_{\mathbb{P} \in \mathcal{F}_{n(m)}(B_x^{(m)})} M_{x,B^{(m)}}(\mathbb{P}, q).$$

Then we claim that Equation 4.41 arises from computing $S(B_x^{(m)}, q)$ in two different ways.

In the first way, we will place m rooks in each column of $B_x^{(m)}$, starting with the leftmost column and working right. In the first column, if the rooks are placed in the x-part, then by how we defined $M_{C,x,B^{(m)}}(C_1, q)$, we will get a total

q-weight of $[x^m]_q$ over all placements of rooks in the x-part. Similarly, if the rooks are placed above the high bar, then we get a total q-weight of $[b_1^m]_q$ over all such placements. So, the q-weight over all possible placements of m rooks in C_1 is $[x^m]_q + [b_1^m]_q$. Since rooks do not cancel to their right in this board, if we place m rooks in C_j, then the total q-weight over all placements of rooks in this column will be $[x^m]_q + [b_j^m]_q$, and thus,

$$S(B_x^{(m)}, q) = \prod_{i=1}^{n} ([x^m]_q + [b_i^m]_q).$$

Now, fix a placement $\mathbb{P} \in \mathcal{F}_{n-k,(m)}(B^{(m)})$. We wish to compute the sum of the weights of all placement $Q \in \mathcal{F}_{n,(m)}(B_x^{(m)})$ such $Q \cap B^{(m)} = \mathbb{P}$. Each such Q arises by placing m rooks below the high bar in each column which does not contain a rook of $\mathbb{P}$. In each such column there will be x^m ways of placing these m rooks, which will give a total q-weight for each column of $[x^m]_q$. As there are k such columns,

$$\begin{aligned} S(B_x^{(m)}, q) &= \textstyle\sum_{k=0}^{n} \sum_{P \in \mathcal{F}_{n-k,(m)}(B^{(m)})} \mu(\mathbb{P}, q)([x^m]_q)^k \\ &= \textstyle\sum_{k=0}^{n} \overline{f}_{n-k,(m)}(B^{(m)}, q)([x^m]_q)^k. \end{aligned}$$

□

Specializing the above theorem to the case where $B = \mathbf{B}_n$, we get the following product formula involving the signless type II q-Stirling numbers:

$$\prod_{i=1}^{n} ([x^m]_q + ([(i-1)^m]_q = \sum_{k=0}^{n} \overline{c}_{n,k}^{x^m}(q)([x^m]_q)^k. \tag{4.42}$$

If we then take Equation (4.42) and replace $[x^m]_q$ with $-[x^m]_q$ and multiply both sides by $(-1)^n$, then we get a product formula involving the $\overline{s}_{n,k}^{x^m}(q)$'s:

$$\prod_{i=1}^{n} ([x^m]_q - [(i-1)^m]_q) = \sum_{k=0}^{n} \overline{s}_{n,k}^{x^m}(q)([x^m]_q)^k. \tag{4.43}$$

By Corollary 4.18, we then get the following product formula involving the type II q-Stirling numbers of the second kind,

$$([x^m]_q)^n = \sum_{k=0}^{n} \overline{S}_{n,k}^{x^m}(q)[x^m]_q([x^m]_q - [1^m]_q) \cdots ([x^m]_q - [(k-1)^m]_q). \tag{4.44}$$

We can give a direct combinatorial proof of (4.44). Suppose that we are given two n-length sequences of nonnegative integers $\mathcal{B}$ and $\mathcal{A}$ and a positive integer m. We will define, for any $\mathbb{P} \in \mathcal{N}_{k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)})$ the quantity

$$\Gamma_{\mathcal{B}^{\mathcal{A},(m)}}(\mathbb{P}, q) = \prod_{i=1}^{k} \gamma_{r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j_i,m)}, q),$$

where the km rooks in $\mathbb{P}$ lie in the columns $C_{j_1}, \ldots, C_{j_k}$, and where the m rooks in column C_{j_i} are denoted by $r^{(j_i,m)} = (r_{(1,j_i)}, \ldots, r_{(m,j_i)})$.

We also set for any $Q \in \mathcal{N}_{n,(m)}^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A},(m)})$

$$\Gamma_{x,\mathcal{B}^{\mathcal{A},(m)}}(Q, q) = \prod_{j=1}^{n} \gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j,m)}, q).$$

We then define $\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r_{(l,j)}, q)$ as follows:

1. If the rooks $(r_{(1,j)}, \ldots, r_{(m,j)})$ lie, respectively, in the cells $c(a_1, 1, j)$, …, $c(a_m, m, j)$ of the $\mathcal{B}$-part of the board, then

$$\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j,m)}, q) = q^{((a_1-1)\cdots(a_m-1))_{b_j}}.$$

2. If the rooks $(r_{(1,j)}, \ldots, r_{(m,j)})$ lie, respectively, in the cells $c_x(a_1, 1, j)$, …, $c_x(a_m, m, j)$ of the x-part of the board, then

$$\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j,m)}, q) = q^{((x-a_1)\cdots(x-a_m))_{b_j}}.$$

3. Suppose the rooks $(r_{(1,j)}, \ldots, r_{(m,j)})$ lie, respectively, in the cells $c(a_1, 1, j)$, …, $c(a_m, m, j)$ of the upper augmented part of the board. If sm rooks have been placed above the high bar in the first $j-1$ columns, then

$$\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j,m)}, q) = q^{(((a_1-b_j-1)\cdots(a_m-b_j-1))_{A_{j-s}}}.$$

Figure 4.18: An example of type II q-counting in the board $\mathcal{B}_x^{\mathcal{A},(m)}$, where $\mathcal{B} = (1,2,4)$, $\mathcal{A} = (1,2,1)$, $x = 3$, and $m = 2$.

4. Suppose the rooks $(r_{(1,j)}, \ldots, r_{(m,j)})$ lie, respectively, in the cells $\tilde{c}(a_1, 1, j)$, …, $\tilde{c}(a_m, m, j)$ of the lower augmented part of the board. If sm rooks have been placed above the high bar in the first $j-1$ columns, then

$$\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j,m)}, q) = -q^{((a_1-1)\cdots(a_m-1))_{A_{j-s}}}.$$

We can see an example of this type of q-counting in Figure 4.18, where $\mathcal{B} = (1,2,4)$, $\mathcal{A} = (1,2,1)$, $x = 3$, and $m = 2$. If Q is the placement shown in the left-hand side of the diagram, then we get a q-weight for Q of

$$\begin{aligned}
\Gamma_{x,\mathcal{B}^{\mathcal{A},(m)}}(Q,q) &= \gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(1,m)}, q)\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(2,m)}, q)\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(3,m)}, q) \\
&= (q^{(00)_1})(q^{(21)_3})(-q^{(12)_3}) \\
&= (q^1)(q^7)(-q^5) \\
&= -q^{13}.
\end{aligned}$$

Theorem 4.20. *Let $x, n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. If $\mathcal{B} = (b_1, \ldots, b_n)$,$\mathcal{A} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, then*

$$\prod_{i=1}^{n}([x^m]_q + [b_i^m]_q) = \sum_{k=0}^{n} \overline{r}^{\mathcal{A}}_{n-k,(m)}(\mathcal{B}^{\mathcal{A},(m)}, q) \prod_{j=1}^{k}([x^m]_q - [A_j]_q), \tag{4.45}$$

where $A_i = a_1 + \cdots + a_i$.

Proof: Let

$$G(\mathcal{B}_x^{\mathcal{A},(m)}, q) = \sum_{Q \in \mathcal{N}^{\mathcal{A}}_{n,(m)}(\mathcal{B}_x^{\mathcal{A},(m)})} \Gamma_{x,\mathcal{B}^{\mathcal{A},(m)}}(Q, q).$$

We then claim that Equation (4.45) arises by computing the sum $G(\mathcal{B}_x^{\mathcal{A},(m)}, q)$ in two different ways.

In the first way, we will place m rooks in each column of $\mathcal{B}_x^{\mathcal{A},(m)}$, starting with the leftmost column and working to the right. In the first column, if we place the m rooks in the x-part of the board, then there are x^m way to do this, and we will get a total q-weight, over all of these x^m possibilities, of $[x^m]_q$. Now if we place the m rooks in the $\mathcal{B}$-part, then there are b_1^m ways to place these rooks, and the total q-weight over these b_1^m placements will be $[b_1^m]_q$. Finally, there are $2(A_1^m)$ ways to place the rooks in the augmented parts of the boards, however, we weight these choices differently. That is, there are A_1^m ways of placing the rooks in the both the upper and lower augmented parts of the board, but the q-weighting over the placements of rooks in the upper augmented part is $[A_1^m]_q$ and the q-weighting over the placements of rooks in the lower augmented part is $-[A_1^m]_q$. Thus, if we consider the $x^m + b_1^m + 2(A_1)^m$ possible placements of m rooks in C_1, we get a total q-weight of $[x^m]_q + [b_1^m]_q + [A_1^m]_q - [A_1^m]_q = [x^m]_q + [b_1^m]_q$ for these placements. When we go to place rooks in the second column of the board, we have two cases.

Case I: Suppose the m rooks in the first column were placed above the high bar. Then, the cells from the a_2-part of the board were cancelled in the second column, and thus, when we go to place m rooks in the second column, we are back in a similar situation as before. That is, we have $x^m + b_2^m + 2(A_1^m)$ ways of placing the rooks. However, there placements give a total q-weight of $[x^m]_q + [b_2^m]_q + [A_1^m]_q - [A_1^m]_q = [x^m]_q + [b_2^m]_q$.

Case II: Suppose that the m rooks placed in the first column were place below the high bar. Then, no cells were cancelled in the second columns, and we will have $x^m + b_2^m + 2(A_2^m)$ ways to place the m rooks. Again though, the total weighting for these placements will be $[x^m]_q + [b_2^m]_q + [A_2^m]_q - [A_2^m]_q = [x^m]_q + [b_2^m]_q$.

In general, suppose we are going to place rooks in the j^{th} column of $\mathcal{B}_x^{\mathcal{A},(m)}$, and suppose that we have placed rooks in s columns above the high bar in the first $j-1$ columns. Then in this column there will be $x^m + b_j^m + 2(A_{j-s}^m)$ ways in which to place the rooks. If we q-count over all of these possible placements however, we see that they will contributes a total q-weight of $[x^m]_q + [b_j^m]_q + [A_{j-s}^m]_q - [A_{j-s}^m]_q = [x^m]_q + [b_j^m]_q$, and so

$$G(\mathcal{B}_x^{\mathcal{A},(m)}, q) = \prod_{i=1}^{n}([x^m]_q + [b_i^m]_q).$$

In the second way of counting G, we first fix a placement $\mathbb{P} \in \mathcal{N}_{n-k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)})$. As in previous proofs, we now wish to compute to sum of the weights of all placements $Q \in \mathcal{N}_{n,(m)}^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A},(m)})$ such $Q \cap \mathcal{B}^{\mathcal{A},(m)} = \mathbb{P}$. Suppose that we label the columns of $\mathcal{B}_x^{\mathcal{A},(m)}$ which contain a rook from $\mathbb{P}$ as $C_{i_1}, \ldots, C_{i_{n-k}}$ and the columns which do not contain a rook of $\mathbb{P}$ as $C_{\tilde{i_1}}, \ldots, C_{\tilde{i_k}}$. One can see that each such Q arises by placing m rooks below the high bar in the columns $C_{\tilde{i_1}}, \ldots, C_{\tilde{i_k}}$. In each of these k columns, there will be x^m ways of placing the rooks in the x-part, and each will contribute a q-weight of $[x^m]_q$. There will also be, in column $\tilde{C}_{i_t}$, A_t^m ways of placing rooks in the lower augmented part, but the total q-weight over these placements will be $-[A_t^m]_q$. By how we defined the q-weight of the rooks placed in column j, $\gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(j,m)}, q)$, we have that

$$\begin{aligned}
\Gamma_{x,\mathcal{B}^{\mathcal{A},(m)}}(Q,q) &= (\textstyle\prod_{j=1}^{n-k} \gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(i_j,m)}, q))(\prod_{j=1}^{k} \gamma_{x,r,\mathcal{B}^{\mathcal{A},(m)}}(r^{(\tilde{i_j},m}, q)) \\
&= \Gamma_{\mathcal{B}^{\mathcal{A},(m)}}(\mathbb{P}, q)(\textstyle\prod_{j=1}^{k}([x^m]_q - [A_j^m]_q)).
\end{aligned}$$

Summing over all possible k and $\mathbb{P}$, we get that

$$G(\mathcal{B}_x^{\mathcal{A},(m)}, q) = \sum_{k=0}^{n} \overline{r}_{n-k,(m)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A},(m)}, q) \prod_{j=1}^{k}([x^m]_q - [A_j^m]_q),$$

which is the desired result. □

We now see that Equation (4.44) follows directly from this theorem.

Figure 4.19: An example of the polyboard $B(p(x))$, with $B = F(1, 2, 3, 5, 5)$ and $p(x) = a_0 + a_1x + a_2x^2$.

4.5 Poly-Stirling Numbers

4.5.1 Notation

In this section we wish to describe another rook model setting which will allows us to give a combinatorial interpretation to the $s_{n,k}^{p(x)}$ and $S_{n,k}^{p(x)}$ for any polynomial in $p(x) \in \mathbb{N}[x]$. Suppose we are given a rook board $B = F(b_1, b_2, \ldots, b_n)$ and a polynomial $p(x) = a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$ where $0 \leq s_1 < s_2 < \cdots < s_y$ and $a_{s_i} \neq 0$ for all $1 \leq i \leq y$. We will then define a set of y m-partition boards $B(p(x)) := \{B^{(s_1)}, B^{(s_2)}, \ldots, B^{(s_y)}\}$. In the case where $s_1 = 0$, $B^{(0)}$ is a degenerate board with n columns of height 0. For such a board, rooks may be placed in any column of $B^{(0)}$, but these rooks will not cancel any cells in $B^{(0)}$. We will call $B(p(x))$ the *polyboard associated with* B *and* $p(x)$, and we will refer to the board $B^{(s_z)}$ as the z^{th} *subboard of* $B(p(x))$. In Figure 4.19, we see an example of a polyboard where $B = F(1, 2, 3, 5, 5)$ and $p(x)$ is of the form $a_0 + a_1x + a_2x^2$. (Note that the coefficients of $p(x)$ are irrelevant when constructing $B(p(x))$.)

Define $C_{(l,j)}^{z}(B(p(x)))$ to be the l^{th} subcolumn of the j^{th} column of $B^{(s_z)}$, and we will refer to the collection of the j^{th} columns of the y boards in $B(p(x))$ to be the j^{th} *column of* $B(p(x))$. If s_z rooks are placed in column $C_{(l,j)}^{z}(B(p(x)))$, then they will be denoted by the s_z-tuple of rooks $r^{(z,j,s_z)} = (r_{(z,1,j)}, r_{(z,2,j)}, \ldots, r_{(z,s_z,j)})$, and if the rook $r = r_{(z,l,j)}$ is placed in the i^{th} row from the bottom of $B^{(s_z)}$, then we say that r lies in the cell $c(z, i, l, j)$. If we consider columns $C_{(l,j)}^{z}(B(p(x)))$ and $C_{(h,k)}^{z}(B(p(x)))$ with $j < k$ ($j > k$), then we say that $C_{(l,j)}^{z}(B(p(x)))$ lies to the left (right) of $C_{(h,k)}^{z}(B(p(x)))$. Finally, suppose that rooks are placed into k columns of the polyboard. We will label the columns which contain rooks, from left to right (using the afore mentioned notion of left and right), with $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$.

4.5.2 Poly-Rook & Poly-File Numbers

Fix a polynomial $p(x) = a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$ where $0 \leq s_1 < s_2 < \cdots < s_y$ and $a_{s_i} \neq 0$ for all $1 \leq i \leq y$ and a Ferrers board $B = F(b_1, b_2, \ldots, b_n)$. Given $B(p(x))$, we shall consider placements where if there is a rook in a subcolumn $C^z_{(l,k)}$ where $s_z > 0$, then there must be rooks in all the subcolums $C^z_{(l,j)}$ for $1 \leq l \leq s_z$ but there are no rooks in the any of j^{th} columns of any of the boards $B^{(s_i)}$ for $i \neq z$. Next we define how rook cancels in $B(p(x))$ by considering a placement $\mathbb{P}$ of rooks in $B(p(x))$.

Case I: $s_1 = 0$ If a rook r is placed in column k of $B^{(s_1)}$, then

(i) it cancels all the cells in each subcolumn of column j for $B^{(s_i)}$ for $i \neq 1$ and

(ii) it cancels the lowest cells in each of the subcolumns of column l in $B^{(s_i)}$ for $i \neq 1$ and $j' > j$ that have not been cancelled by a rook which lies in a column to the left of column j in one of the boards $B^{(s_1)}, \ldots, B^{(s_z)}$.

Case II: $s_1 > 0$ If a rook r is placed in a subcolumn of column j of $B^{(s_1)}$, then

(i) it cancels the cells in $B^{(s_1)}$ in exactly the same way that any rook cancels in a non-attacking rook placement in $B^{(s_1)}$,

(ii) it cancels all the cells in each subcolumn of column j in each $B^{(s_z)}$ with $z \neq i$, and

(iii) it cancels the lowest cells in each of the subcolumns of column j' in $B^{(s_i)}$ for $i \neq 1$ and $j' > j$ that have not been cancelled by a rook which lies in a column to the left of column j in one of the boards $B^{(s_1)}, \ldots, B^{(s_z)}$.

Case III: s_i **for** $i > 1$ If a rook r is placed in a subcolumn of column j of $B^{(s_i)}$, then

(i) it cancels the cells in $B^{(s_1)}$ in exactly the same way that any rook cancels in a non-attacking rook placement in $B^{(s_1)}$,

(ii) it cancels all the cells in each subcolumn of column j for $B^{(s_z)}$ for $z \neq i$,

Figure 4.20: An example a non-attacking rook placement in the polyboard $B(p(x))$, with $B = F(1, 2, 3, 5, 5)$ and $p(x) = a_0 + a_1x + a_2x^2$.

and

(iii) it cancels the lowest cells in each of the subcolumns of column j' in $B^{(s_i)}$ for $i \neq 1$ and $j' > j$ that have not been cancelled by a rook which lies in a column to the left of column j in one of the boards $B^{(s_1)}, \ldots, B^{(s_z)}$.

An example of this cancellation can be seen in Figure 4.20, where a placement of rooks in three columns of the polyboard $B(p(x))$ is shown, with $B = (1, 2, 3, 5, 5)$ and $p(x) = a_0 + a_1x + a_2x^2$. Here we have placed a "1" in all cells cancelled by the rook in column 1 of the second board, a "2" in all cells cancelled by a the two rooks in column 3 of the third board, and a "3" in the cells cancelled by the rook in column 4 of the first board.

Let $\mathcal{N}_{k,p(x)}(B(p(x)))$ denote the set of all colored rook placements in $B(p(x))$ such that

(a) for all $1 \leq i \leq z$, if there is a rook in a subcolumn of column j in a board $B^{(s_i)}$, then there is exactly one rook in each subcolumn of column j in board $B^{(s_i)}$ and there are no rooks in column j on any other the boards $B^{(s_r)}$ for $r \neq i$,

(b) no rook lies in a cell which is cancelled by another rook,

(c) there are exactly k columns which contain a rook,

(d) if a rook is placed in board $B^{(s_i)}$, then it is colored with one of a_{s_i} distinct colors, $c_1^i, \ldots, c_{a_{s_i}}^i$, and

(e) if a rook placed in $C_{(l,j)}^z$ is colored with color c, then every rook placed in $C_{(t,j)}^z$, for $t = 1, 2, \ldots, s_z$, is colored with c.

We then define the k^{th} *poly-rook number of* $B(p(x))$ to be

$$r_{k,p(x)}(B(p(x))) = |\mathcal{N}_{k,p(x)}(B(p(x)))|.$$

We also consider file placements in the polyboard, where in a file placement, the rooks in $r^{(z,j,s_z)}$ will cancel the cells directly below themselves in the board

Figure 4.21: An example a file rook placement in the polyboard $B(p(x))$, with $B = F(1,2,3,5,5)$ and $p(x) = a_0 + a_1x + a_2x^2$.

$B^{(s_z)}$ as well as all of the cells of the j^{th} column in every board $B^{(s_w)}$ with $w \neq z$. An example of this type of placement and cancellation can be seen in Figure 4.21.

Let $\mathcal{F}_{k,p(x)}(B(p(x)))$ to be the set of colored file placements such that
(a) for all $1 \leq i \leq z$, if there is a rook in a subcolumn of column j in a board $B^{(s_i)}$, then there is exactly one rook in each subcolumn of column j in board $B^{(s_i)}$ and there are no rooks in column j on any other the boards $B^{(s_r)}$ for $r \neq i$,
(b) there are exactly k columns which contain a rook,
(c) if a rook is placed in board $B^{(s_i)}$, then it is colored with one of a_{s_i} distinct colors, $c_1^i, \ldots, c_{a_{s_i}}^i$, and
(d) if a rook placed in $C_{(l,j)}^z$ is colored with color c, then every rook placed in $C_{(t,j)}^z$, for $t = 1, 2, \ldots, s_z$, is colored with c.

We then define the k^{th} *poly-file number of* B to be

$$f_{k,p(x)}(B(p(x))) = |\mathcal{F}_{k,p(x)}(B(p(x)))|.$$

Next we consider file placements in the y-tuple of boards

$$B_x(p(x)) = \{B_x^{(s_1)}, B_x^{(s_2)}, \ldots, B_x^{(s_y)}\}$$

for some $x \in \mathbb{N}$. We shall refer to the *upper part of* $B_x(p(x))$ as the collection of upper parts of the boards $B_x^{(s_1)}, B_x^{(s_2)}, \ldots, B_x^{(s_y)}$ and the *x-part of* $B_x(p(x))$ as the collection of upper parts of the boards $B_x^{(s_1)}, B_x^{(s_2)}, \ldots, B_x^{(s_y)}$ We then say that the upper part of $B_x(p(x))$ is separated from the x-part of $B_x(p(x))$ by the *bar of* $B_x(p(x))$. We also make the following cancellation rules for this set of boards:

1. If r is placed above the high bar in $C_{(l,j)}^z(B_x(p(x)))$ for some z, then r cancels the cells directly below itself but above the high bar in $B_x^{(s_z)}$ as well as every cell above the high bar in the j^{th} column of $B_x^{(s_w)}$ for every $w \neq z$.

Figure 4.22: An example a file rook placement in the 3-tuple $B_x(p(x))$, with $B = F(1,2,3,5,5)$ and $p(x) = a_0 + a_1x + a_2x^2$.

2. If r is placed in the x-part of $C^z_{(l,j)}(B_x(p(x)))$ for some z, then r cancels the cells directly below itself in $B^{(s_z)}$ as well as cancelling all of the cells in the x-part of column j in $B_x^{(s_w)}$ for every $w \neq z$.

An example of the above cancellation can be seen in Figure 4.22, and we let $\mathcal{F}_{n,p(x)}(B_x(p(x)))$ denote the set of placements of colored rooks in n columns of $B_x(p(x))$.

Let $\mathcal{F}_{n,p(x)}(B(p(x)))$ be the set of colored file placements such that
(a) for all $1 \leq i \leq z$, if there is a rook in a subcolumn of column j in a board $B^{(s_i)}$, then there is exactly one rook in each subcolumn of column j in board $B^{(s_i)}$ and there are no rooks in column j on any other the boards $B_x^{(s_r)}$ for $r \neq i$,
(b) for all $1 \leq i \leq z$, if there is a rook in a subcolumn of column j in the x-part of board $B^{(s_i)}$, then there is exactly one rook in each subcolumn of column j in the x-part of board $B^{(s_i)}$ and there are no rooks in column j on any other the boards $B_x^{(s_r)}$ for $r \neq i$,
(c) there are exactly n columns which contain a rook,
(d) if a rook is placed in board $B_x^{(s_i)}$, then it is colored with one of a_{s_i} distinct colors, $c^i_1, \ldots, c^i_{a_{s_i}}$, and
(e) if a rook placed in $C^z_{(l,j)}$ is colored with color c, then every rook placed in $C^z_{(t,j)}$, for $t = 1, 2, \ldots, s_z$, is colored with c.

Theorem 4.21. *Suppose $x, n \in \mathbb{N}$ and $p(x) = a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$ with $a_{s_i} \neq 0$ for every i. If $B = F(b_1, b_2, \ldots, b_n)$ is any rook board, then*

$$\prod_{i=1}^{n}(p(x) + p(b_i)) = \sum_{k=0}^{n} f_{n-k,p(x)}(B(p(x)))(p(x))^k. \tag{4.46}$$

Proof: Given a rook board $B = F(b_1, b_2, \ldots, b_n)$ and $p(x) \in \mathbb{N}[x]$, we shall show that Equation (4.46) represents two ways to count $|\mathcal{F}_{n,p(x)}(B_x(p(x)))|$. We first consider the number of ways that we can place rooks in each column, starting with the leftmost column and working to the right. In the first column of $B(p(x))$ there will be $x^{s_1}+x^{s_2}+\cdots+x^{s_y}$ ways to place rooks below the high bar, and there will be $b_1^{s_1} + b_1^{s_2} + \cdots + b_1^{s_y}$ ways to place rooks above the high bar. However, the total number of placements is different since we are considering colored placements, and thus the total number of colored placements of rooks below the bar is $a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} = p(x)$ and the total number of placements above the bar is $a_{s_1}b_1^{s_1} + a_{s_2}b_1^{s_2} + \cdots + a_{s_y}b_1^{s_y} = p(b_1)$. So, the total number of placements in the first column of $B_x(p(x))$ is $p(x) + p(b_1)$. In general, in the j^{th} column of the $B_x(p(x))$, there will be $p(x)$ total colored placements below in the x-parts and $p(b_j)$ colored placements above the bar, and thus

$$|\mathcal{F}_{n,p(x)}(B_x(p(x)))| = \prod_{i=1}^{n}(p(x) + p(b_i)).$$

Next, suppose that we first fix a placement $\mathbb{P} \in \mathcal{F}_{n-k,p(x)}(B(p(x))))$ above the bar. We claim that there are $(p(x))^k$ ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_{n,p(x)}(B_x(p(x)))$ such that $Q \cap B(p(x)) = \mathbb{P}$. That is, we want to count the number of ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_{n,p(x)}(B_x(p(x)))$ by placing additional colored rooks below the bar in those columns which contain no rooks from $\mathbb{P}$. Here, we see that for each empty column, there are exactly $p(x)$ ways to place colored rooks in that column. As there are k such columns, we have

$$\begin{aligned}
|\mathcal{F}_{n,p(x)}(B_x(p(x)))| &= \sum_{k=0}^{n} \sum_{\mathbb{P}\in\mathcal{F}_{n-k,p(x)}(B(p(x)))} (p(x))^k \\
&= \sum_{k=0}^{n}(p(x))^k \sum_{\mathbb{P}\in\mathcal{F}_{n-k,p(x)}(B(p(x)))} 1 \\
&= \sum_{k=0}^{n}(p(x))^k f_{n-k,p(x)}(B(p(x))),
\end{aligned}$$

Figure 4.23: The board $\mathcal{B}^{\mathcal{A}}(p(x))$, with $\mathcal{B} = (1,1,2,2)$, $\mathcal{A} = (2,1,2,1)$, and $p(x) = a_0 + a_1x + a_2x^2$.

which is the desired result. □

Now consider the sequences $\mathcal{B} = (b_1, \dots, b_n) \in \mathbb{N}^n$ and $\mathcal{A} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$. Given $x \in \mathbb{N}$, we will define $\mathcal{B}^{\mathcal{A},(m)}$ as before. If $p(x) = a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$, then we set $\mathcal{B}^{\mathcal{A}}(p(x)) = \{\mathcal{B}^{\mathcal{A},(s_1)}, \mathcal{B}^{\mathcal{A},(s_2)}, \dots, \mathcal{B}^{\mathcal{A},(s_y)}\}$. We will refer to the set of the $\mathcal{B}$-parts of the y boards as the *$\mathcal{B}$-part of* $\mathcal{B}^{\mathcal{A}}(p(x))$, the set of augmented parts as the *augmented part of* $\mathcal{B}^{\mathcal{A}}(p(x))$, and the set of a_s-parts as the *a_s-part of* $\mathcal{B}^{\mathcal{A}}(p(x))$. An example of $\mathcal{B}^{\mathcal{A}}(p(x))$ can be seen in Figure 4.23, where $\mathcal{B} = (1,1,2,2)$, $\mathcal{A} = (2,1,2,1)$, and $p(x) = a_0 + a_1x + a_2x^2$.

Next we define how rook cancels in $\mathcal{B}^{\mathcal{A}}(p(x))$.

Case I: $s_1 = 0$ If a rook r is placed in column j of $\mathcal{B}^{\mathcal{A},(s_1)}$, then, for each $j' > j$, r cancels all the cells in a_s-part of column j' of the boards $B^{(\mathcal{A},s_2)}, \dots, B^{\mathcal{A},(s_z)}$ where s is the highest index t such that no cells in a_t-part of the boards $B^{\mathcal{A},(s_2)}, \dots, B^{\mathcal{A},(s_z)}$ have been cancelled by a rook which lies to left of column j.

Case II: $s_1 > 0$ If a rook r is placed in column j of $\mathcal{B}^{\mathcal{A},(s_1)}$, then, for each $j' > j$, r cancels all the cells in a_s-part of column j' of the boards $B^{\mathcal{A},(s_1)}, \dots, B^{\mathcal{A},(s_z)}$ where s is the highest index t such that no cells in a_t-part of the boards $B^{\mathcal{A},(s_1)}, \dots, B^{\mathcal{A},(s_z)}$ have been cancelled by a rook which lies to left of column j.

Case III: s_i **for** $i > 1$ If a rook r is placed in column j of $\mathcal{B}^{\mathcal{A},(s_i)}$, then, for each $j' > j$, r cancels all the cells in a_s-part of column j' of the boards $B^{\mathcal{A},(s_1)}$, … ,$B^{\mathcal{A},(s_z)}$ where s is the highest index t such that no cells in a_t-part of the boards $B^{\mathcal{A},(s_1)}, \dots, B^{\mathcal{A},(s_z)}$ have been cancelled by a rook which lies to left of column j.

Figure 4.24: A placement of rooks in the board $\mathcal{B}^{\mathcal{A}}(p(x))$, with $\mathcal{B} = (1,1,2,2)$, $\mathcal{A} = (2,1,2,1)$, and $p(x) = a_0 + a_1x + a_2x^2$.

An example of this cancellation can be seen in Figure 4.24, where cancellation arising from the rook in $c(2,1,1,1)$ is denoted by a "$\bullet$" and the cells cancelled by the rook in $c(1,1,1,3)$ are marked with an "$*$".

We will let $\mathcal{N}_{k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))$ denote the set of colored rook placements in $\mathcal{B}^{\mathcal{A}}(p(x))$ such that

(a) for all $1 \leq i \leq z$, if there is a rook in a subcolumn of column j of the $\mathcal{B}$-part (augmented part) in a board $B^{\mathcal{A},(s_i)}$, then there is exactly one rook in each subcolumn of column j of the $\mathcal{B}$-part (augmented part) in board $B^{\mathcal{A},(s_i)}$ and there are no rooks in column j on any other the boards $B^{\mathcal{A},(s_r)}$ for $r \neq i$,
(b) no rook lies in a cell which is cancelled by another rook,
(c) there are exactly k columns which contain a rook,
(d) if a rook is placed in board $B^{\mathcal{A},(s_i)}$, then it is colored with one of a_{s_i} distinct colors, $c_1^i, \ldots, c_{a_{s_i}}^i$, and
(e) if a rook placed in $C_{(l,j)}^z$ is colored with color c, then every rook placed in $C_{(t,j)}^z$, for $t = 1, 2, \ldots, s_z$, is colored with c.

We will then define

$$r_{k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x))) = |\mathcal{N}_{k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x)))|$$

to be the k^{th} *type* $\mathcal{A}$ *poly-rook number associated with* $\mathcal{B}$, $\mathcal{A}$, *and* $p(x)$.

We will also define $\mathcal{B}_x^{\mathcal{A}}(p(x)) = \{\mathcal{B}_x^{\mathcal{A},(s_1)}, \mathcal{B}_x^{\mathcal{A},(s_2)}, \ldots, \mathcal{B}_x^{\mathcal{A},(s_y)}\}$. For each $1 \leq z \leq y$, we use the terms a_s-part, $\mathcal{B}$-part, x-part, upper augmented part, lower augmented part, high bar, and low bar as in previous sections when referring to each subboard, and if we refer to the *lower augmented part of* $\mathcal{B}_x^{\mathcal{A}}(p(x))$, then we are talking about the set of lower augmented parts of the $\mathcal{B}^{\mathcal{A},(s_z)}$. An example of

Figure 4.25: The board $\mathcal{B}_x^{\mathcal{A}}(p(x))$, with $\mathcal{B} = (1,1,2,2)$, $\mathcal{A} = (2,1,2,1)$, and $p(x) = a_0 + a_1x + a_2x^2$.

$\mathcal{B}_x^{\mathcal{A}}(p(x))$ is illustrated in Figure 4.25.

Next we define how rook cancells in $\mathcal{B}_x^{\mathcal{A}}(p(x))$.

Case I: $s_1 = 0$ If a rook r is placed in column j of $\mathcal{B}_x^{\mathcal{A},(s_1)}$, then, for each $j' > j$, r cancels all the cells in a_s-part of column j' in both the upper and lower augmented parts of the boards $B_x^{(s_2)}, \ldots, B_x^{(s_z)}$ where s is the highest index t such that no cells in a_t-part in the upper and lower augmented parts of the boards $B_x^{(s_2)}, \ldots, B_x^{(s_z)}$ have been cancelled by a rook which lies to left of column j.

Case II: $s_1 > 0$ If a rook r is placed in column j above the upper bar in $\mathcal{B}_x^{\mathcal{A},(s_1)}$, then, for each $j' > j$, r cancels all the cells in a_s-part of column j' in both the upper and lower augmented parts of the boards $B_x^{(s_1)}, \ldots, B_x^{(s_z)}$ where s is the highest index t such that no cells in a_t-part in the upper and lower augmented parts of the boards $B_x^{(s_1)}, \ldots, B_x^{(s_z)}$ have been cancelled by a rook which lies to left of column j. If a rook r is placed in column j below the upper bar in $\mathcal{B}_x^{\mathcal{A},(s_1)}$, then it does not cancel any squares.

Case III: s_i **for** $i > 1$ If a rook r is placed in column k above the upper bar in $\mathcal{B}_x^{\mathcal{A},(s_i)}$, then, for each $j' > j$, r cancels all the cells in a_s-part of column j' in both the upper and lower augmented parts of the boards $B_x^{(s_1)}, \ldots, B_x^{(s_z)}$ where s is the highest index t such that no cells in a_t-part in the upper and lower augmented parts of the boards $B_x^{(s_1)}, \ldots, B_x^{(s_z)}$ have been cancelled by a rook which lies to left of column k. If a rook r is placed in column j below the upper bar in $\mathcal{B}_x^{\mathcal{A},(s_i)}$, then it does not cancel any squares.

An example of this type of cancellation can be seen in Figure 4.26.

We let $\mathcal{N}_{n,p(x)}^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}(p(x)))$ denote the set of all colored rook placements in $B(p(x))$ such that

Figure 4.26: A placement of rooks in the board $\mathcal{B}_x^{\mathcal{A}}(p(x))$, with $\mathcal{B} = (1,1,2,2)$, $\mathcal{A} = (2,1,2,1)$, and $p(x) = a_0 + a_1 x + a_2 x^2$.

(a) for all $1 \leq i \leq z$, if there is a rook in a subcolumn of column j in a board $B_x^{\mathcal{A},(s_i)}$ in the $\mathcal{B}$-part (x-part, upper augemented part, lower augmented part), then there is exactly one rook in each subcolumn of column j in board $B^{\mathcal{A},(s_i)}$in the $\mathcal{B}$-part (x-part, upper augemented part, lower augmented part) and there are no rooks in column j on any other the boards $B_x^{\mathcal{A},(s_r)}$ for $r \neq i$,
(b) no rook lies in a cell which is cancelled by another rook,
(c) there are exactly n columns which contain a rook,
(d) if a rook is placed in board $B_x^{\mathcal{A},(s_i)}$, then it is colored with one of a_{s_i} distinct colors, $c_1^i, \ldots, c_{a_{s_i}}^i$, and
(e) if a rook placed in $C_{(l,j)}^z$ is colored with color c, then every rook placed in $C_{(t,j)}^z$, for $t = 1, 2, \ldots, s_z$, is colored with c.

Theorem 4.22. *Given $x, n \in \mathbb{N}$ and $p(x) = a_{s_1}x^{s_1} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$ with $a_{s_i} \neq 0$ for every i, let $\mathcal{B} = (b_1, b_2, \ldots, b_n)$ and $\mathcal{A} = (a_1, a_2, \ldots, a_n)$ be two sequences in $\mathbb{N}^n$. Then*

$$\prod_{i=1}^{n}(p(x) + p(b_i)) = \sum_{k=0}^{n} r_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x))) \prod_{j=1}^{k}(p(x) - p(A_j)), \tag{4.47}$$

where $A_i = a_1 + a_2 + \cdots + a_i$.

Proof: Define, for any $Q \in \mathcal{N}_{n,p(x)}^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}(p(x)))$,

$$W_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(Q) = \prod_{j=1}^{n} w_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(r^{(z,j,s_z)}),$$

where $r^{(z,j,s_z)}$ represents the s_z rooks which are placed in column j and we define $w_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(r^{(z,j,s_z)})$ by

1. $w_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(r^{(z,j,s_z)}) = -1$ if the rooks in $r^{(z,j,s_z)}$ lie in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}(p(x))$ and

2. $w_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(r^{(z,j,s_z)}) = 1$ otherwise.

If $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))$, and if the rooks are in columns $C_{i_1}, C_{i_2}, \dots, C_{i_k}$, then set

$$W_{\mathcal{B}^{\mathcal{A}}(p(x))}(\mathbb{P}) = \prod_{j=1}^{k} w_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(r^{(z,i_j,s_z)}).$$

If we define

$$S(\mathcal{B}^{\mathcal{A}}_x(p(x))) = \sum_{\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{n,p(x)}(\mathcal{B}^{\mathcal{A}}_x(p(x)))} W_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(\mathbb{P}),$$

then we claim that Theorem 4.26 represents computing $S(\mathcal{B}^{\mathcal{A}}_x(p(x)))$ in two different ways.

Suppose we first wish to place rooks in every column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ by starting with the leftmost column and working to the right. If we consider non-colored rook placements, then in the first column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, there are $x^{s_1} + \cdots + x^{s_y}$ ways to place rooks in the x-part, $b_1^{s_1} + \cdots + b_1^{s_y}$ cells to place rooks in the $\mathcal{B}$-part, and $2(A_1^{s_1} + \cdots + A_1^{s_y})$ cells to place rooks in the augmented parts. Thus, there will be $p(x) + p(b_1) + 2p(A_1)$ ways to place colored rooks in the first column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, but these are weighted as $p(x) + p(b_1) + p(A_1) - p(A_1) = p(x) + p(b_1)$. When we place rooks in the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ we have two cases.

Case I: If in the first column we placed the rooks in above the high bar, then the cells from the a_2-part of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ were cancelled in the upper and lower augmented parts of the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$. Thus, the total number of colored rook placements in the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ is $p(x)+p(b_2)+2p(A_1)$, but the total weight of these placements is $p(x) + p(b_2) + p(A_1) - p(A_1) = p(x) + p(b_2)$.

Case II: If in the first column the rooks were placed below the high bar in $\mathcal{B}^{\mathcal{A}}_x(p(x))$, then no cells were cancelled in the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, and thus there are $p(x) + p(b_2) + 2p(A_2)$ possible colored rook placements in that column, but these again come with a total weight of $p(x) + p(b_2)$.

Suppose that in the first $j-1$ columns of our board, we have placed rooks in s columns of the upper part of $\mathcal{B}_x^{\mathcal{A}}(p(x))$. Then in the j^{th} column of $\mathcal{B}_x^{\mathcal{A}}(p(x))$, the cells from the $a_j, \ldots, a_{j-s+1}$-parts gave been cancelled in both the upper and lower augmented parts of that column. We will then have $p(x)+p(b_j)+2p(A_{j-s})$ possible colored rook placements in the j^{th} column of $\mathcal{B}_x^{\mathcal{A}}(p(x))$, but these will have a total weight of $p(x)+p(b_j)+p(A_{j-s})-p(A_{j-s})=p(x)+p(b_j)$, and so,

$$S(\mathcal{B}_x^{\mathcal{A}}(p(x))) = \prod_{i=1}^{n}(p(x)+p(b_j)).$$

Now suppose that we fix a rook placement $\mathbb{P} \in \mathcal{N}_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x)))$. We now wish to compute

$$\sum_{\substack{Q \in \mathcal{N}_{n,p(x)}(\mathcal{B}_x^{\mathcal{A}}(p(x))) \\ Q \cap \mathcal{B}^{\mathcal{A}}(p(x)) = \mathbb{P}}} W_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(Q).$$

Each Q will arise from placing rooks below the high bar in each column of $\mathcal{B}_x^{\mathcal{A}}(p(x))$ which does not contain a rook of $\mathbb{P}$. There are k such columns, and we will begin placing the rooks in these columns by starting with the leftmost available column and working to the right.

In the first available column, there will be $p(x)+p(A_1)$ colored rook placements possible, the total weight of which is $p(x)-p(A_1)$. In fact, in the k^{th} available column of $\mathcal{B}_x^{\mathcal{A}}(p(x))$ below the high bar, we will have $p(x)+p(A_k)$ possible placements of colored rooks, but these will have a total weight of $p(x)-p(A_k)$. By how we define $W_{x,\mathcal{B}^{\mathcal{A}}(p(x))}$ and $W_{\mathcal{B}^{\mathcal{A}}(p(x))}$, we have

$$\sum_{\substack{Q \in \mathcal{N}_{n,p(x)}^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}(p(x))) \\ Q \cap \mathcal{B}^{\mathcal{A}}(p(x)) = \mathbb{P}}} W_{x,\mathcal{B}^{\mathcal{A}}(p(x))}(Q) = W_{\mathcal{B}^{\mathcal{A}}(p(x))}(\mathbb{P})\ (p(x)-p(A_1))\cdots(p(x)-p(A_k)).$$

Thus,

$$\begin{aligned} S(\mathcal{B}_x^{\mathcal{A}}(p(x))) &= \textstyle\sum_{k=0}^{n} \sum_{\mathbb{P}\in\mathcal{N}_{n-k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))} (p(x)-p(A_1))\cdots(p(x)-p(A_k)) \\ &= \textstyle\sum_{k=0}^{n} r^{\mathcal{A}}_{n-k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))\,(p(x)-p(A_1))\cdots(p(x)-p(A_k)). \quad \square \end{aligned}$$

4.5.3 Poly-Stirling Numbers of the First & Second Kind

Consider again Equations (4.1), (4.3), and (4.2), which are restated here. The *Poly-Stirling numbers of the first kind*, $s_{n,k}^{p(x)}$, are defined by the following recursions:

$$\begin{aligned} &s_{0,0}^{p(x)} = 1 \text{ and } s_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &s_{n+1,k}^{p(x)} = s_{n,k-1}^{p(x)} - p(n)s_{n,k}^{p(x)} \text{ if } 0 \le k \le n+1 \text{ and } n \ge 0. \end{aligned}$$

We now define $s_{n,k}^{p(x)} = (-1)^{n-k}c_{n,k}^{p(x)}$. Thus, the integers $c_{n,k}^{p(x)}$, called the *signless poly-Stirling numbers of the first kind*, satisfy the recursions:

$$\begin{aligned} &c_{0,0}^{p(x)} = 1 \text{ and } c_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &c_{n+1,k}^{p(x)} = c_{n,k-1}^{p(x)} + p(n)c_{n,k}^{p(x)} \text{ if } 0 \le k \le n+1 \text{ and } n \ge 0. \end{aligned}$$

Finally, we will denote the *Poly-Stirling numbers of the second kind* by $S_{n,k}^{p(x)}$, and we will define them by the recursions:

$$\begin{aligned} &S_{0,0}^{p(x)} = 1 \text{ and } S_{n,k}^{p(x)} = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &S_{n+1,k}^{p(x)} = S_{n,k-1}^{p(x)} + p(k)S_{n,k}^{p(x)} \text{ if } 0 \le k \le n+1 \text{ and } n \ge 0. \end{aligned}$$

Theorem 4.23. *For $p(x) \in \mathbb{N}[x]$, if $B = \boldsymbol{B}_n$, the staircase board with n columns, then*

$$c_{n,k}^{p(x)} = f_{n-k,p(x)}(B(p(x))).$$

Figure 4.27: The seventeen rook placements which correspond to $c_{3,1}^{1+x^2}$.

Proof: We will denote the polyboard associated with $\mathbf{B}_n$ and $p(x)$ by $\mathbf{B}_n(p(x))$. If $n, k < 0$ or $k > n$, then $f_{n-k,p(x)}(\mathbf{B}_n(p(x))) = 0$, and if $n = 0$ we will say that $f_{0,p(x)}(\emptyset) = 0$. Thus, the initial conditions have been satisfied. Suppose $n > 0$.

We will now proceed by induction on the number of columns of the board $\mathbf{B}_n$. When $n = 1$ we have $f_{1,p(x)}(\mathbf{B}_1(p(x))) = p(0) = c_{1,0}^{p(x)}$ and $f_{0,p(x)}(\mathbf{B}_1(p(x))) = 1 = c_{1,1}^{p(x)}$. So assume that $n > 1$ and $f_{n-k,p(x)}(\mathbf{B}_n(p(x))) = c_{n,k}^{p(x)}$ for $0 \leq k \leq n$. By our definition, $f_{n+1-k,p(x)}(\mathbf{B}_{n+1}(p(x)))$ will be equal to the number of colored file placements into $n + 1 - k$ columns of $\mathbf{B}_{n+1}(p(x))$. These placements will either have no rooks placed in the last column of $\mathbf{B}_{n+1}(p(x))$ or there will be rooks placed in the last column. Those placements which have no rooks in the last column of $\mathbf{B}_{n+1}(p(x))$ are counted by $f_{n+1-k,p(x)}(\mathbf{B}_n(p(x)))$, which is equal to $c_{n,k-1}^{p(x)}$ by our induction hypothesis. Similarly, if there are rooks placed in the last column of $\mathbf{B}_{n+1}(p(x))$, then there are only $n-k$ columns of the first n which contain rooks, and we have assumed that those are counted by $f_{n-k,p(x)}(\mathbf{B}_n(p(x))) = c_{n,k}^{p(x)}$. Now we need to extend these placements in $\mathbf{B}_n(p(x))$ to placements in the board $\mathbf{B}_{n+1}(p(x))$. Since none of the rooks in the first n column of $\mathbf{B}_{n+1}(p(x))$ cancel to their right, there will be $p(n)$ ways to place colored rooks in rightmost column of $\mathbf{B}_{n+1}(p(x))$. Thus,

$$\begin{aligned} f_{n+1-k,p(x)}(\mathbf{B}_{n+1}(p(x))) &= f_{n+1-k,p(x)}(\mathbf{B}_n(p(x))) + p(n) f_{n-k,p(x)}(\mathbf{B}_n(p(x))) \\ &= c_{n,k-1}^{p(x)} + p(n) c_{n,k}^{p(x)} \\ &= c_{n+1,k}^{p(x)}. \end{aligned}$$

□

This gives us the following corollary of Theorem 4.21

$$\prod_{i=1}^{n} (p(x) + p(i-1)) = \sum_{k=0}^{n} c_{n,k}^{p(x)} (p(x))^k. \tag{4.48}$$

Figure 4.28: The eighteen rook placements which correspond to $S_{4,3}^{1+x^2}$.

If in Equation (4.48) we replace $p(x)$ with $-p(x)$ and multiply both sides by $(-1)^n$, we get a product formula which involves the $s_{n,k}^{p(x)}$:

$$\prod_{i=1}^{n}(p(x)-p(i-1))=\sum_{k=0}^{n}s_{n,k}^{p(x)}(p(x))^k. \tag{4.49}$$

From the Milne Inversion Theorem 4.1, we also know that we must have the following product formula:

$$(p(x))^n=\sum_{k=0}^{n}S_{n,k}^{p(x)}(p(x))(p(x)-p(1))\cdots(p(x)-p(k-1)); \tag{4.50}$$

however this formula will be a direct result of Theorem 4.24

Theorem 4.24. *For any* $p(x)\in\mathbb{N}[x]$*, if* $\mathcal{B}=(0,0,\ldots,0)$ *and* $\mathcal{A}=(0,1,\ldots,1)$*, then*

$$S_{n,k}^{p(x)}=r_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x))).$$

Proof Sketch: In the usual way, we need only show that for the given $\mathcal{B}$ and $\mathcal{A}$, the numbers defined by $S_{n,k}^{p(x)}$ satisfy the same initial conditions and recursions as the $r_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x)))$. □

4.6 Type I Q-Analogues

4.6.1 Type I Q-Counting in Polyboards

Suppose that we are given a placement $\mathbb{P}\in\mathcal{F}_{k,p(x)}(B(p(x)))$. Then for each set of rooks $r=r^{(z,j,s_z)}$ we define the q-weight of r by

$$g(r,q)=q^{\alpha(r)},$$

Figure 4.29: q-counting in the board $B(p(x))$, with the same placement as in Figure 4.21. Here the q-weight is $(1)(q)(1) = q$.

where $\alpha(r)$ is the number of cells in $B(p(x))$ that lie directly above the rooks in r. We then define the q-weight of $\mathbb{P}$ to be

$$G(\mathbb{P}, q) = \prod_{r \in \mathbb{P}} g(r, q).$$

An illustration of this type of q-counting can be seen in Figure 4.29, where the same placement is used as in Figure 4.21. Here we see that

$$\begin{aligned} G(\mathbb{P}, q) &= g(r^{(2,1,1)}, q)\ g(r^{(3,3,2)}, q)\ g(r^{(1,4,1)}, q) \\ &= (1)(q)(1) \\ &= q. \end{aligned}$$

We then define the k^{th} *type I* q*-poly-file numbers* to be

$$f_{k,p(x)}(B(p(x)), q) = \sum_{\mathbb{P} \in \mathcal{F}_{k,p(x)}(B(p(x)))} G(\mathbb{P}, q). \tag{4.51}$$

Now suppose that we are given a placement $\mathbb{P} \in \mathcal{F}_{n,p(x)}(\mathcal{B}_x(p(x)))$. We will define, for each rook $r \in \mathbb{P}$, the q-weight of r by

$$g_x(r, q) = q^{\alpha_x(r)},$$

where

1. $\alpha_x(r)$ is the number if uncancelled cells which lie directly above r if r is not in the x-part of $\mathcal{B}_x(p(x))$, and
2. $\alpha_x(r)$ is the number if uncancelled cells which lie directly above r but below the bar if r is in the x-part of $\mathcal{B}_x(p(x))$.

Figure 4.30: q-counting in the board $B_x(p(x))$, with the same placement as in Figure 4.22. Here the q-weight is $(1)(1)(q)(1)(q^5) = q^6$.

The q-weight of $\mathbb{P}$ is then defined to be

$$G_x(\mathbb{P}, q) = \prod_{r \in \mathbb{P}} g_x(r, q).$$

This q-counting in the board $\mathcal{B}_x(p(x))$ is pictured in Figure 4.30, where the placement shown has q-weight

$$\begin{aligned} G_x(\mathbb{P}, q) &= g_x(r^{(2,1,1)}, q)\, g_x(r^{(1,2,1)}, q)\, g_x(r^{(3,3,2)}, q)\, g_x(r^{(1,4,1)}, q)\, g_x(r^{(3,5,2)}, q) \\ &= (1)(1)(q)(1)(q^5) \\ &= q^6. \end{aligned}$$

Theorem 4.25. *Suppose* $x, n \in \mathbb{N}$ *and* $p(x) = a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$ *with* $a_{s_i} \neq 0$ *for each* i*. If* $B = F(b_1, b_2, \ldots, b_n)$ *is any rook board, then*

$$\prod_{i=1}^{n} (p([x]_q) + p([b_i]_q)) = \sum_{k=0}^{n} f_{n-k,p(x)}(B(p(x)), q)\ (p([x]_q))^k. \tag{4.52}$$

Proof: Given a rook board $B = F(b_1, b_2, \ldots, b_n)$, we shall show that (4.52) represents two ways to count $S_q(\mathcal{B}_x(p(x)))$, where

$$S_q(\mathcal{B}_x(p(x)) = \sum_{\mathbb{P} \in \mathcal{F}_{n,p(x)}(\mathcal{B}_x(p(x)))} G_x(\mathbb{P}, q).$$

We first consider the number of ways that we can place rooks in each column, starting with the leftmost column and working to the right. In the first column of $B(p(x))$ there will be $x^{s_1} + x^{s_2} + \cdots + x^{s_y}$ ways to place rooks below the high bar, and there will be $b_1^{s_1} + b_1^{s_2} + \cdots + b_1^{s_y}$ ways to place rooks above the high bar. However, the total number of placements is different since we are considering colored placements, and thus the total number of colored placements of rooks below the bar is $a_{s_1}x^{s_1} + a_{s_2}x^{s_2} + \cdots + a_{s_y}x^{s_y} = p(x)$. When we sum the q-weights of these placements, we get a total q-weight of $p([x]_q)$. Similarly, the total number of placements above the bar is $a_{s_1}b_1^{s_1} + a_{s_2}b_1^{s_2} + \cdots + a_{s_y}b_1^{s_y} = p(b_1)$, and the sum of the q-weights of all of these placements is $p([b_1]_q)$. Thus, the total q-weight over all placements in the first column of $B_x(p(x))$ is $p([x]_q) + p([b_1]_q)$. In general, in the j^{th} column of the $B_x(p(x))$, there will be $p(x)$ total colored placements below in the x-parts and $p(b_j)$ colored placements above the bar, and the total q-weight over all of these placements will be $p([x]_q) + p([b_j]_q)$, so

$$S_q(\mathcal{B}_x(p(x)) = \prod_{i=1}^{n}(p([x]_q) + p([b_i]_q)).$$

Next, suppose that we first fix a file placement $\mathbb{P} \in \mathcal{F}_{n-k,p(x)}(B(p(x))))$. Then the q-weight of $\mathbb{P}$ is $G_x(\mathbb{P}, q)$. We know that there are $(p(x))^k$ ways to extend $\mathbb{P}$ to a placement $Q \in \mathcal{F}_{n,p(x)}(B_x(p(x)))$ such that $Q \cap B(p(x)) = \mathbb{P}$, however we want to q-count each of these extensions. We know that for each available column, there we will get a total q-count of $p([x]_q)$. By how we defined $G_x(Q, q)$, we get

$$\sum_{\substack{Q \in \mathcal{F}_{n,p(x)}(\mathcal{B}_x(p(x))) \\ Q \cap B(p(x)) = \mathbb{P}}} G_x(Q, q) = G_x(\mathbb{P}, q)(p([x]_q)^k).$$

Thus,

$$\begin{aligned} S_q(\mathcal{B}_x(p(x)) &= \sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{F}_{n-k,p(x)}(B(p(x)))} G_x(\mathbb{P}, q)(p([x]_q))^k \\ &= \sum_{k=0}^{n} f_{n-k,p(x)}(B(p(x)), q)\ (p([x]_q))^k, \end{aligned}$$

which is the desired result. □

Let $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))$, and suppose that $r \in \mathbb{P}$. Then we can define the q-weight of r to be

Figure 4.31: q-counting in the board $\mathcal{B}^{\mathcal{A}}(p(x))$, with the same placement as in Figure 4.24. Here the q-weight is $(q)(1) = q$.

$$h(r, q) = q^{\beta^{\mathcal{A}}(r)},$$

where

1. $\beta^{\mathcal{A}}(r)$ is the number of uncancelled cells directly above the rooks in r if r is in the augmented part of $\mathcal{B}^{\mathcal{A}}(p(x))$, and
2. $\beta^{\mathcal{A}}(r)$ is the number of uncancelled cells above the rooks in r in the $\mathcal{B}$-part of $\mathcal{B}^{\mathcal{A}}(p(x))$ if r is in the $\mathcal{B}$-part of $\mathcal{B}^{\mathcal{A}}(p(x))$.

We then define the q-weight of $\mathbb{P}$ to be

$$H(\mathbb{P}, q) = \prod_{r \in \mathbb{P}} h(r, q).$$

In Figure 4.31, we see that the placement shown, which is the identical placement to Figure 4.24, has a q-weight of

$$\begin{aligned} H(\mathbb{P}, q) &= h(r^{(2,1,1)}, q)\, h(r^{(1,3,1)}, q) \\ &= (1)(q) \\ &= q. \end{aligned}$$

We then define the k^{th} *type I $q\mathcal{A}$-poly-rook number of* $\mathcal{B}^{\mathcal{A}}(p(x))$ to be

$$r^{\mathcal{A}}_{k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)), q) = \sum_{\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))} H(\mathbb{P}, q). \tag{4.53}$$

Now suppose that we are given a placement $\mathbb{P} \in \mathcal{N}^{\mathcal{A}}_{n,p(x)}(\mathcal{B}^{\mathcal{A}}_x(p(x)))$. We will set, for each $r \in \mathbb{P}$, the q-weight of r to be

Figure 4.32: q-counting in the board $\mathcal{B}_x^{\mathcal{A}}(p(x))$, with the same placement as in Figure 4.26. Here the q-weight is $(q)(q)(1)(-q^3) = -q^5$.

$$h_x(r,q) = q^{\beta_x^{\mathcal{A}}(r)},$$

where

1. $\beta_x^{\mathcal{A}}(r)$ is equal to $\beta^{\mathcal{A}}(r)$ if r is above the high bar in $\mathcal{B}_x^{\mathcal{A}}(p(x))$,
2. $\beta_x^{\mathcal{A}}(r)$ is equal to the number of uncancelled cells directly above the rooks in r but below the high bar if r is in the x-part of $\mathcal{B}_x^{\mathcal{A}}(p(x))$, and
3. $\beta_x^{\mathcal{A}}(r)$ is equal to the negative of the number of uncancelled cell directly above a rook in r but below the low bar if r is in the lower augmented part of $\mathcal{B}_x^{\mathcal{A}}(p(x))$.

Using this weighting scheme we set the q-weight of $\mathbb{P}$ to be

$$H_x(\mathbb{P},q) = \prod_{r\in\mathbb{P}} h_x(r,q).$$

This type of q-counting in the board $\mathcal{B}_x^{\mathcal{A}}(p(x))$ can be seen in Figure 4.32, where the placement shown has q-weight

$$\begin{aligned} H_x(\mathbb{P},q) &= h_x(r^{(2,1,1)},q)\, h_x(r^{(2,2,1)},q)\, h_x(r^{(1,3,1)},q)\, h_x(r^{(3,4,2)},q) \\ &= (q)(q)(1)(-q^3) \\ &= -q^5. \end{aligned}$$

Theorem 4.26. *Given $x, n \in \mathbb{N}$ and $p(x) = a_{s_1}x^{s_1} + \cdots + a_{s_y}x^{s_y} \in \mathbb{N}[x]$ with $a_{s_i} \neq 0$ for every i, let $\mathcal{B} = (b_1, b_2, \ldots, b_n)$ and $\mathcal{A} = (a_1, a_2, \ldots, a_n)$ be two sequences in $\mathbb{N}^n$. Then*

$$\prod_{i=1}^{n}(p([x]_q) + p([b_i]_q)) = \sum_{k=0}^{n} r^{\mathcal{A}}_{n-k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)),q) \prod_{j=1}^{k}(p([x]_q) - p([A_j]_q)). \quad (4.54)$$

Proof: Define, for any $Q \in \mathcal{N}^{\mathcal{A}}_{n,p(x)}(\mathcal{B}^{\mathcal{A}}_x(p(x)))$,

$$S(\mathcal{B}^{\mathcal{A}}_x(p(x)), q) = \sum_{Q \in \mathcal{N}^{\mathcal{A}}_{n,p(x)}(\mathcal{B}^{\mathcal{A}}_x(p(x)))} H_x(Q, q).$$

Then we claim that Theorem 4.26 represents computing $S(\mathcal{B}^{\mathcal{A}}_x(p(x)), q)$ in two different ways.

Suppose we first wish to place rooks in every column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ by starting with the leftmost column and working to the right. Then in the first column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, there are $x^{s_1} + \cdots + x^{s_Y}$ cells to place rooks in the x-part, $b_1^{s_1} + \cdots + b_1^{s_y}$ cells to place rooks in the $\mathcal{B}$-part, and $2(A_1^{s_1} + \cdots + A_1^{s_y})$ cells to place rooks in the augmented parts. Thus, there are $p(x) + p(b_1) + 2p(A_1)$ ways to place colored rooks in the first column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, however, the total q-weight of these placements is $p([x]_q) + p([b_1]_q) + p([A_1]_q) - p([A_1]_q) = p([x]_q) + p([b_1]_q)$. When we go to place rooks in the second column, we have two cases.

<u>Case I:</u> If in the first column we placed the rooks in above the high bar, then the cells from the a_2-part of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ we cancelled in the upper and lower augmented parts of of the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$. Thus, the total number of colored rook placements in the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$ is $p(x) + p(b_2) + 2p(A_1)$, but the total weight of these placements is $p([x]_q) + p([b_2]_q) + p([A_1]_q) - p([A_1]_q) = p([x]_q) + p([b_2]_q)$.

<u>Case II:</u> If in the first column the rooks were placed below the high bar, then no cells were cancelled in the second column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, and thus there are $p(x) + p(b_2) + 2p(A_2)$, but these again come with a total weight of $p([x]_q) + p([b_2]_q) + p([A_2]_q) - p([A_2]_q) = p([x]_q) + p([b_2]_q)$.

Now suppose that in the first $j - 1$ columns of our board, we have placed rooks in s columns of the upper part of $\mathcal{B}^{\mathcal{A}}_x(p(x))$. Then in the j^{th} column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, the cells from the $a_j, \ldots, a_{j-s+1}$-parts gave been cancelled in both the upper and lower augmented parts of that column. We will then have $p(x) + p(b_j) + 2p(A_{j-s})$ colored rook placements in the j^{th} column of $\mathcal{B}^{\mathcal{A}}_x(p(x))$, but these will have a total weight of $p([x]_q) + p([b_j]_q) + p([A_{j-s}]_q) - p([A_{j-s}]_q) = p([x]_q) + p([b_j]_q)$, and so,

$$S(\mathcal{B}_x^{\mathcal{A}}(p(x))) = \prod_{i=1}^{n} (p([x]_q) + p([b_i]_q)).$$

Now suppose that we fix a rook placement $\mathbb{P} \in \mathcal{N}_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x)))$. Then the q-weight of $\mathbb{P}$ is $H(\mathbb{P}, q)$, and we now wish to compute

$$\sum_{\substack{\mathcal{Q} \in \mathcal{N}_{n,p(x)}(\mathcal{B}_x^{\mathcal{A}}(p(x))) \\ \mathcal{Q} \cap \mathcal{B}^{\mathcal{A}}(p(x)) = \mathbb{P}}} H_x(Q, q).$$

Each of these Q's will arise from placing rooks below the high bar in each column of $\mathcal{B}_x^{\mathcal{A}}(p(x))$ which does not contain a rook of $\mathbb{P}$. There are k such columns, and we will begin placing the rooks in these columns by starting with the leftmost available column and working to the right. In the first available column, there will be $p(x) + p(A_1)$ colored rook placements possible, the total q-weight of which is $p([x]_q) - p([A_1]_q)$. In fact, in the w^{th} available column of $\mathcal{B}_x^{\mathcal{A}}(p(x))$ below the high bar, we will have $p(x) + p(A_w)$ possible placements of colored rooks, but these will have a total $q$weight of $p([x]_q) - p([A_w]_q)$. By how we define $H_x(Q, q)$ and $H(\mathbb{P}, q)$ we have

$$\sum_{\substack{Q \in \mathcal{N}_{n,p(x)}^{\mathcal{A}}(\mathcal{B}_x^{\mathcal{A}}(p(x))) \\ Q \cap \mathcal{B}^{\mathcal{A}}(p(x)) = \mathbb{P}}} H_x(Q, q) = H(\mathbb{P}, q)(p([x]_q) - p([A_j]_q)) \cdots (p([x]_q) - p([A_k]_q)).$$

Thus,

$$\begin{aligned} S(\mathcal{B}_x^{\mathcal{A}}(p(x)), q) &= \textstyle\sum_{k=0}^{n} \sum_{\mathbb{P} \in \mathcal{N}_{n-k,p(x)}(\mathcal{B}^{\mathcal{A}}(p(x)))} H(\mathbb{P}, q) \prod_{j=1}^{k} (p([x]_q) - p([A_j]_q)) \\ &= \textstyle\sum_{k=0}^{n} r_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x)), q) \prod_{j=1}^{k} (p([x]_q) - p([A_j]_q)). \end{aligned}$$

□

4.6.2 Type I Q-Poly-Stirling Numbers

In this section we will study the polynomials defined by the recursions

$$\begin{aligned} &S^{p(x)}_{0,0}(q) = 1 \text{ and } S^{p(x)}_{n,k}(q) = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &S^{p(x)}_{n+1,k}(q) = S^{p(x)}_{n,k-1}(q) + p([k]_q)S^{p(x)}_{n,k}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.55}$$

We will call these numbers the *type I q-poly Stirling numbers of the second kind*.

As before, the inverses of these numbers satisfy the recursion

$$\begin{aligned} &s^{p(x)}_{0,0}(q) = 1 \text{ and } s^{p(x)}_{n,k}(q) = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &s^{p(x)}_{n+1,k}(q) = s^{p(x)}_{n,k-1}(q) - p([n]_q)s^{p(x)}_{n,k}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.56}$$

We will call these numbers the *type I q-poly Stirling numbers of the first kind*. If we now replace $s^{x^m}_{n,k}(q)$ with $(-1)^{(n-k)}c^{x^m}_{n,k}(q)$, then we have the numbers which satisfy the recursion

$$\begin{aligned} &c^{p(x)}_{0,0}(q) = 1 \text{ and } c^{p(x)}_{n,k}(q) = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &c^{p(x)}_{n+1,k}(q) = c^{p(x)}_{n,k-1}(q) + p([n]_q)c^{p(x)}_{n,k}(q) \text{ if } 0 \leq k \leq n+1 \text{ and } n \geq 0. \end{aligned} \tag{4.57}$$

and we will call these numbers the *signless type I q-poly Stirling numbers of the first kind*.

Theorem 4.27. *Let $n \in \mathbb{N}$ and $p(x) \in \mathbb{N}[x]$. If $B = \boldsymbol{B}_n$, then, for every $0 \leq k \leq n$,*

$$c^{p(x)}_{n,k}(q) = f_{n-k,p(x)}(\boldsymbol{B}_n(p(x)), q). \tag{4.58}$$

Proof: We will denote the polyboard associated with $\mathbf{B}_n$ and $p(x)$ by $\mathbf{B}_n(p(x))$. If $n, k < 0$ or $k > n$, then $f_{n-k,p(x)}(\mathbf{B}_n(p(x)), q) = 0$, and if $n = 0$ we will say that $f_{0,p(x)}(\emptyset, q) = 0$. Thus, the initial conditions have been satisfied. Suppose $n > 0$.

We will now proceed by induction on the number of columns of the board $\mathbf{B}_n$. When $n = 1$ we have $f_{1,p(x)}(\mathbf{B}_1(p(x)), q) = p(0) = c^{p(x)}_{1,0}(q)$ and also that

$f_{0,p(x)}(\mathbf{B}_1(p(x)),q) = 1 = c^{p(x)}_{1,1}(q)$. So assume $n > 1$ and $f_{n-k,p(x)}(\mathbf{B}_n(p(x)),q) = c^{p(x)}_{n,k}(q)$ for $0 \leq k \leq n$. By our definition, $f_{n+1-k,p(x)}(\mathbf{B}_{n+1}(p(x)),q)$ will be equal to the total q-weight over all colored file placements into $n+1-k$ columns of $\mathbf{B}_{n+1}(p(x))$. These placements will either have no rooks placed in the last column of $\mathbf{B}_{n+1}(p(x))$ or there will be rooks placed in the last column. Those placements which have no rooks in the last column of $\mathbf{B}_{n+1}(p(x))$ are q-counted by $f_{n+1-k,p(x)}(\mathbf{B}_n(p(x)),q)$, which is equal to $c^{p(x)}_{n,k-1}(q)$ by our induction hypothesis. Similarly, if there are rooks placed in the last column of $\mathbf{B}_{n+1}(p(x))$, then there are only $n-k$ columns of the first n which contain rooks, and we have assumed that those are counted by $f_{n-k,p(x)}(\mathbf{B}_n(p(x)),q) = c^{p(x)}_{n,k}(q)$. Now we need to extend these placements in $\mathbf{B}_n(p(x))$ to placements in the board $\mathbf{B}_{n+1}(p(x))$. Since none of the rooks in the first n column of $\mathbf{B}_{n+1}(p(x))$ cancel to their right, there will be $p(n)$ ways to place colored rooks in rightmost column of $\mathbf{B}_{n+1}(p(x))$, which will contribute a q-weight of $p([n]_q)$ to the total q-count in the poly-staircase board. Thus,

$$\begin{aligned} f_{n+1-k,p(x)}(\mathbf{B}_{n+1}(p(x)),q) &= f_{n+1-k,p(x)}(\mathbf{B}_n(p(x)),q) \\ &\quad + p([n]_q)f_{n-k,p(x)}(\mathbf{B}_n(p(x)),q) \\ &= c^{p(x)}_{n,k-1}(q) + p([n]_q)c^{p(x)}_{n,k}(q) \\ &= c^{p(x)}_{n+1,k}(q). \end{aligned}$$

□

We now have the product formula

$$\prod_{i=1}^{n}(p([x]_q) + p([i-1]_q)) = \sum_{k=0}^{n} c^{p(x)}_{n,k}(q)(p([x]_q))^k, \tag{4.59}$$

and if we then replace $p([x]_q)$ in the above equation with $-p([x]_q)$ and multiply both sides by $(-1)^n$, then we get

$$\prod_{i=1}^{n}(p([x]_q) - p([i-1]_q)) = \sum_{k=0}^{n} s_{n,k}^{p(x)}(q)(p([x]_q))^k. \tag{4.60}$$

Now, we can apply the Milne Inversion Theorem 4.1 to show that the matrices $||S_{n,k}^{p(x)}(q)||$ and $||s_{n,k}^{p(x)}(q)||$ are inverses of one another, which also leads to the product formula

$$(p([x]_q)^n = \sum_{k=0}^{n} S_{n,k}^{p(x)}(q) \prod_{j=1}^{k}(p([x]_q) - p([j-1]_q)), \tag{4.61}$$

although this formula arises as a corollary to Theorem 4.28.

Theorem 4.28. *Let* $n \in \mathbb{N}$ *and* $p(x) \in \mathbb{N}[x]$*. If* $\mathcal{B} = (0, 0, \ldots, 0)$ *and* $\mathcal{A} = (0, 1, \ldots, 1)$*, then, for every* $0 \leq k \leq n$*,*

$$S_{n,k}^{p(x)}(q) = r_{n-k,p(x)}^{\mathcal{A}}(\mathcal{B}^{\mathcal{A}}(p(x)), q). \tag{4.62}$$

Proof Sketch: We need only show that these two polynomials satisfy the same recursions and initial conditions. □

4.7 Type II Q-Analogues

As we did in Section 3.3.5, we can express the polynomial $[p(x)]_q$, called the *type II* q*-analogue of* $p(x)$, in various forms. Recall that for nonnegative integers x and a with $x \geq a$, we have the identity $[x+a]_q = [x]_q + q^x[a]_q$. As an example, consider the type II q-analogue of the polynomial $p(x) = x^2 + 2x$. We can then rewrite $[p(x)]_q$ as

$$[x^3 + 2x + 4]_q = q^{2x}[x^2]_q + [2x]_q.$$

We can see that if we are given a polyboard $B(x^2 + 2x)$, then for the the $q^{2x}[x^2]_q$ portion of this q-analogue, we can just cosnider a type II q-weighting in

Figure 4.33: An example of how we would q-count cells to acheive a factor of $[2x]_q$.

the board $B^{(2)}$ with every cell weighted with an extra factor of q^{2x}. However, when we look at the $[2x]_q$ portion of $[p(x)]_q$, then we must do something slightly different. Here , we will need a total q-count of $1+q+\cdots+q^{2x-1}$, and we have x cells to account for that q-weight. Thus, we will weight the cells of $B^{(1)}$, from top to bottom, with the weights $1+q$, q^2+q^3, …, $q^{x-2}+q^{x-1}$. An example of this can be seen in Figure 4.33.

In this way, we can now q-count both non-attacking and file placements of rooks in the polyboard in a similar fashion to how we counted type II q-analogues in x^m-boards. We see that by appropriately weighting the cells in each of the boards of $B(p(x))$ with extra factors of $p_z(x)$ (as we did in Chapter 3) or weighting cell with polynomials in q which are not simply powers of q, we can generate type II q-poly rook and q-poly-file numbers as well as their corresponding product formulas. Moreover, when the board $B = \mathbf{B}_n$, we can relate these numbers to type II q-poly-Stirling numbers, which are defined in the following section.

4.8 Type II Q-Poly-Stirling Numbers

Consider the numbers defined by the recursions

$$\begin{aligned}&\overline{S}^{p(x)}_{0,0}(q) = 1 \text{ and } \overline{S}^{p(x)}_{n,k}(q) = 0 \text{ if } k<0 \text{ or } k>n \text{ and}\\ &\overline{S}^{p(x)}_{n+1,k}(q) = \overline{S}^{p(x)}_{n,k-1}(q) + [p(k)]_q \overline{S}^{p(x)}_{n,k}(q) \text{ if } 0 \le k \le n+1 \text{ and } n \ge 0.\end{aligned} \tag{4.63}$$

We will call these numbers the *type II q-poly Stirling numbers of the second kind.*

As before, the inverses of these numbers satisfy the recursions

$$\begin{aligned}&\overline{s}^{p(x)}_{0,0}(q) = 1 \text{ and } \overline{s}^{p(x)}_{n,k}(q) = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &\overline{s}^{p(x)}_{n+1,k}(q) = \overline{s}^{p(x)}_{n,k-1}(q) - [p(n)]_q \overline{s}^{p(x)}_{n,k}(q) \text{ if } 0 \le k \le n+1 \text{ and } n \ge 0.\end{aligned} \tag{4.64}$$

We will call these numbers the *type II q-poly Stirling numbers of the first kind*. If we now replace $\overline{s}^{x^m}_{n,k}(q)$ with $(-1)^{(n-k)}\overline{c}^{x^m}_{n,k}(q)$, then we have the numbers which satisfy the recursion

$$\begin{aligned}&\overline{c}^{p(x)}_{0,0}(q) = 1 \text{ and } \overline{c}^{p(x)}_{n,k}(q) = 0 \text{ if } k < 0 \text{ or } k > n \text{ and} \\ &\overline{c}^{p(x)}_{n+1,k}(q) = \overline{c}^{p(x)}_{n,k-1}(q) + [p(n)]_q \overline{c}^{p(x)}_{n,k}(q) \text{ if } 0 \le k \le n+1 \text{ and } n \ge 0,\end{aligned} \tag{4.65}$$

and we will call these numbers the *signless type II q-poly Stirling numbers of the first kind*.

By using methods similar to those in previous sections, for the appropriate method of q-counting in $\mathbf{B}_{n,x}(p(x))$, we can show that

$$\prod_{i=1}^{n}([p(x)]_q + [p(i-1)]_q) = \sum_{k=0}^{n} \overline{c}^{p(x)}_{n,k}(q)([p(x)]_q)^k. \tag{4.66}$$

If we then replace $[p(x)]_q$ in the above equation with $-[p(x)]_q$ and multiply both sides by $(-1)^n$, then we get

$$\prod_{i=1}^{n}([p(x)]_q - [p(i-1)]_q) = \sum_{k=0}^{n} \overline{s}^{p(x)}_{n,k}(q)([p(x)]_q)^k. \tag{4.67}$$

Now, we can again apply the Milne Inversion Theorem 4.1 to show that the matrices $||\overline{S}^{p(x)}_{n,k}(q)||$ and $||\overline{s}^{p(x)}_{n,k}(q)||$ are inverses of one another, which also leads to the product formula

$$([p(x)]_q)^n = \sum_{k=0}^{n} \overline{S}^{p(x)}_{n,k}(q) \prod_{j=1}^{k}([p(x)]_q - [p(j-1)]_q). \tag{4.68}$$

This formula can also be proved by type II q-counting in the board $\mathcal{B}^{\mathcal{A}}_x(p(x))$ when $\mathcal{B} = (0, 0, \ldots, 0)$ and $\mathcal{A} = (0, 1, \ldots, 1)$.

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