Handout on Homomorphisms & Factor Groups

**Definition 0.1.** Let $G$ and $G'$ be groups. The function $f : G \to G'$ is a group homomorphism if $f(ab) = f(a)f(b)$ for every $a, b \in G$.

**Theorem 0.2.** Let $G$ and $G'$ be groups with identities $e$ and $e'$, respectively. If $f : G \to G'$ is a homomorphism, then the following are all true.

i. $f(e) = e'$

ii. For any $n \in \mathbb{Z}$ and $x \in G$, $f(x^n) = [f(x)]^n$.

iii. For any $x \in G$ such that $|x| < \infty$, $||f(x)|| |x|$.

**Definition 0.3.** Let $f : G \to G'$ be a group homomorphism. Then the kernel of $f$ is $\text{Ker} f = \{x \in G \mid f(x) = e'\}$.

**Theorem 0.4.** Let $G$ and $G'$ be groups with identities $e$ and $e'$, respectively. If $f : G \to G'$ is a homomorphism with $H \leq G$, then the following are all true.

i. $\text{Ker} f \leq G$.

ii. For any $x, y \in G$, $f(x) = f(y)$ if and only if $x\text{Ker} f = y\text{Ker} f$.

iii. If $f(x) = z$, then $f^{-1}(z) = \{a \in G \mid f(a) = z\} = x\text{Ker} f$.

iv. $f(H) = \{f(h) \mid h \in H\} \leq G'$.

v. If $H$ is cyclic, then $f(H)$ is cyclic.

vi. If $H$ is Abelian, then $f(H)$ is Abelian.

vii. If $H \triangleleft G$, then $f(H) \triangleleft G'$.

viii. If $|\text{Ker} f| = n$, then $f$ is an $n$-to-1 mapping.

ix. If $|H| = n$, then $|f(H)||n$.

x. If $K \leq G'$, then $f^{-1}(K) = \{a \in G \mid f(a) \in K\} \leq G$.

xi. If $K \triangleleft G'$, then $f^{-1}(K) \triangleleft G$.

**Definition 0.5.** Let $G$ be a group and suppose $H \leq G$. We say that $H$ is a normal subgroup of $G$, denoted by $H \triangleleft G$, if $xH = Hx$ for every $x \in G$. This is equivalent to showing that $xHx^{-1} \subseteq H$ for every $x \in G$.

**Theorem 0.6.** Let $f : G \to G'$ be a group homomorphism. Then $\text{Ker} f \triangleleft G$.

**Definition 0.7.** Let $G$ be a group and let $H \triangleleft G$. Then $G/H = \{xH \mid x \in G\}$ is called the factor (or quotient) group of $G$ by $H$.

**Theorem 0.8.** Let $G$ be a group with $H \leq G$. Then $G/H$ is a group if and only if $H \triangleleft G$. The operation of this group is $(aH)(bH) = (ab)H$.

**Remark 0.9.** The previous theorem is the main reason we care about normal subgroups. That is, we only make factor groups with normal subgroups, but in this case, we know that we always get a group. For example, the previous results give that $G/(\text{Ker} f)$ is a group for any homomorphism, $f$, for which $G$ is the domain.
Theorem 0.10 (Cauchy’s Theorem for Finite Abelian Groups). Let $G$ be a finite Abelian group such that $p$ is a prime divisor of the order of $G$. Then $G$ has an element of order $p$.

Corollary 0.11. Let $G$ be a finite Abelian group of order $p_1 p_2 \cdots p_k$ for distinct primes $p_1, p_2, \ldots, p_k$. Then $G$ is cyclic.

Theorem 0.12 (Fundamental Homomorphism Theorem, a.k.a., First Isomorphism Theorem). Let $f : G \to G'$ be a group homomorphism. Then$$G/(\text{Ker } f) \cong f(G).$$

Definition 0.13. Let $G$ be a group and let $H \triangleleft G$. The mapping $\phi : G \to G/N$ given by $\phi(x) = xN$ is called the natural homomorphism from $G$ to $G/N$.

Corollary 0.14 (Second Isomorphism Theorem). Let $G$ be a group with $H \leq G$ and $N \triangleleft G$. Then$$HN/N \cong H/(H \cap N).$$

Corollary 0.15 (Third Isomorphism Theorem). Let $G$ be a group with $H, N \leq G$ and $N \leq H$. Then$$G/H \cong (G/N)/(H/N).$$