Problem 1: Suppose $a, n \in \mathbb{N}$ and let $d = \text{gcd}(a, n)$.

(i) Show that the equation $ax \equiv 1 \text{ mod } n$ has a solution if and only if $d = 1$

(ii) Show that $U(n) = \{k \mid 0 < k < n \text{ and } \gcd(k, n) = 1\}$ is a group under multiplication modulo $n$.

(iii) Show that the set $\mathbb{Z}_p^*$ is a group under multiplication modulo $p$ if and only if $p$ is prime.

Problem 2: Let $G$ denote the set $\{A_0, A_1, A_2, A_3, \ldots\} \subseteq M_n(\mathbb{R})$ under matrix multiplication, where for each $i \geq 0$, $\det(A_i) = 2^i$. If we further suppose that for every $i, j \geq 0$, $A_iA_j = A_{i+j}$, either prove that $G$ is a group or explicitly state why it cannot be a group.

Problem 3: Let $A = \{x + iy \in \mathbb{C} \mid x = 0 \text{ or } y = 0\}$. For any $a, b, \in A$, define $a \ast b = \sqrt{a^2 + b^2}$. Prove that $A$ is a group under the operation $\ast$ or explicitly state why it cannot be a group.

Problem 4: Define the operation $\ast$ on $\mathbb{Q}^+$ by $a \ast b = \frac{ab}{2}$ for any $a, b \in \mathbb{Q}^+$. Prove that $(\mathbb{Q}^+, \ast)$ is a group under the operation $\ast$ or explicitly state why it cannot be a group.

*Bonus Problem #2*: Let $S = \mathbb{R} - \{-1\}$, and define $\ast$ on $S$ by $a \ast b = a + b + ab$ for any $a, b, \in S$. Prove that $(S, \ast)$ is a group under the operation $\ast$ or explicitly state why it cannot be a group.