Supplementary Problems for HW #6

**Problem 1:** Let $S_n$ denote the $n$-th symmetric group. Prove that if $H \leq S_n$, then either all of the permutations of $H$ are even or exactly half of them are even. Use this result to determine the order of the alternating group, $A_n$, which is the set of all even permutations of $S_n$.

**Problem 4:** A card-shuffling machine always rearranges cards in the same way relative to the order in which they are given to it. All of the hearts, arranged in order from ace to king, were put into the machine, shuffled, and then the cards were put into the machine again to be shuffled. If the cards emerged in the order 10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7 after this second shuffle, what was their order after the first shuffle?

**Problem 3:** Show that the mapping $f : \langle \mathbb{C}, + \rangle \to \langle \mathbb{C}, + \rangle$ given by $a + bi \mapsto a - bi$ is an automorphism. What does this automorphism do to the complex plane, geometrically speaking? Show that $f$ also has the property that $f(xy) = f(x)f(y)$ for any $x, y \in \mathbb{C}$.

**Problem 4:** Let $f, g : \langle a \rangle \to G$ be isomorphisms such that $f(a) = g(a)$. Prove that $f(x) = g(x)$ for all $x \in \langle a \rangle$.

**Problem 5:** Consider the elements $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \in SL(2, \mathbb{R})$. Find $|A|$, $|B|$, and $|AB|$.

*Bonus Problem 5:* Let $G$ be a group. Suppose that $a, b \in G$ such that $a$ and $b$ commute and their orders are relatively prime. Prove that $|ab| = |a||b|$.

*Bonus Problem 6:* Suppose $G$ and $H$ are finite groups and let 

$G_k = \{x \in G \mid |x| = k\}$ and $H_k = \{x \in H \mid |x| = k\}$.

If $|G_k| = |H_k|$ for every possible value of $k$, must it be the case that $G \cong H$?