MATH 3362-1 **Modern Algebra I** **Spring 2014**

Handout on The Fundamental Theorem of Finite Abelian Groups

Theorem 0.1 (Fundamental Theorem of Finite Abelian Groups)**.** *Every finite Abelian group is a direct product of cyclic groups of prime power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.*

Example 0.2. Suppose we know that G is an Abelian group of order $200 = 2^3 \cdot 5^2$. Then G is isomorphic to one of the following groups:

$$\mathbb{Z}_8 \times \mathbb{Z}_{25},\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25},$$
$$\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5,\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5.$$

Definition 0.3. Let G be a group and suppose $H, K \leq G$. We say that G is an *internal direct product* of H and K, written as $G = H \otimes K$ if $G = HK$ and $H \cap K = \{e\}$.

Remark 0.4. If $G = H \otimes K$, then it turns out that $H, K \triangleleft G$. Moreover, $hk = kh$ for every $h \in H$ and $k \in K$, even though the elements of H (and similarly, K) may not commute with each other.

Remark 0.5. We have essentially made G from two of its subgroups. Moreover, if $G = H \otimes K$, then the previous theorem give that G/H and G/K are both groups. Unsurprisingly, we can show that $G/H \cong K$ and $G/K \cong H$, that is, the algebra we want to happen does happen in this scenario.

Theorem 0.6. *Suppose $G = \langle a \rangle \otimes \langle b \rangle$ is a group such that $|a| = m$ and $|b| = n$. Then $G \cong \mathbb{Z}_m \times \mathbb{Z}_n$.*

Lemma 1: Let G be a finite Abelian group of order $p^n m$, where $n \in \mathbb{N}$, p is prime, and $\gcd(p, m) = 1$. Then $G = H_p \otimes K$, where $H_p = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$. Moreover, $|H_p| = p^n$.

How Does Lemma 1 Help? Suppose G is an Abelian group such that $|G| = p_1^{a_1} p_2^{a_2} \cdots p_j^{a_j}$. Then we know that

$$\begin{aligned} G &= H_{p_1} \otimes K, \text{ where } |H_{p_1}| = p_1^{a_1} \text{ and } |K| = p_2^{a_2} \cdots p_j^{a_j} \\ &= H_{p_1} \otimes H_{p_2} \otimes K', \text{ where } |H_{p_1}| = p_1^{a_1}, |H_{p_2}| = p_2^{a_2}, \text{ and } |K| = p_3^{a_3} \cdots p_j^{a_j} \\ &\vdots \\ &= H_{p_1} \otimes H_{p_2} \otimes \cdots \otimes H_{p_j}, \text{ where } |H_{p_m}| = p_m^{a_m} \text{ for } 1 \leq m \leq j. \end{aligned}$$

Lemma 2: Let G be an Abelian group of order p^n, where $n \in \mathbb{N}$ and p is prime. If $a \in G$ has maximal order, then $G = \langle a \rangle \otimes K$ for some $K \leq G$.

How Does Lemma 2 Help? This tells us how to further break up our groups of prime power orders. It also allows us to prove Lemma 3.

Lemma 3: Any Abelian group of prime power order is an internal direct product of cyclic groups.

How Does Lemma 3 Help? Once we have that $G = H_{p_1} \otimes H_{p_2} \otimes \cdots \otimes H_{p_j}$ from Lemma 1, we then know that each $H_{p_\ell} = \langle a_{\ell_1} \rangle \otimes \langle a_{\ell_2} \rangle \otimes \cdots \langle a_{\ell_k} \rangle$.

Lemma 4: Let G be an Abelian Group and define $G^n = \{x^n \mid x \in G\}$. Then $G^n \leq G$. Moreover, if p is a prime such that $p||G|$, then $G^p \neq G$.

How Does Lemma 4 Help? We use this in the proof of Lemma 5.

Lemma 5: Suppose that G is an Abelian group of order p^n, where $n \in \mathbb{N}$ and p is prime. If $G = H_{p_1} \otimes H_{p_2} \otimes \cdots \otimes H_{p_w}$ and $G = K_{p_1} \otimes K_{p_2} \otimes \cdots \otimes K_{p_z}$, where each H_{p_i} and K_{p_j} is a non-trivial, cyclic subgroup of G such that $|H_{p_1}| \geq |H_{p_2}| \geq \cdots \geq |H_{p_w}|$ and $|K_{p_1}| \geq |K_{p_2}| \geq \cdots \geq |K_{p_z}|$, then $w = z$ and $|H_{p_r}| = |K_{p_r}|$ for every $1 \leq r \leq w$.

How Does Lemma 5 Help? This gives us the "uniquely determined" part of the theorem.