Handout on The Fundamental Theorem of Finite Abelian Groups

**Theorem 0.1** (Fundamental Theorem of Finite Abelian Groups). Every finite Abelian group is a direct product of cyclic groups of prime power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.

**Example 0.2.** Suppose we know that $G$ is an Abelian group of order $200 = 2^3 \cdot 5^2$. Then $G$ is isomorphic to one of the following groups:

- $\mathbb{Z}_8 \times \mathbb{Z}_{25}$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$,
- $\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

**Definition 0.3.** Let $G$ be a group and suppose $H, K \leq G$. We say that $G$ is an internal direct product of $H$ and $K$, written as $G = H \otimes K$ if $G = HK$ and $H \cap K = \{e\}$.

**Remark 0.4.** If $G = H \otimes K$, then it turns out that $H, K \triangleleft G$. Moreover, $hk = kh$ for every $h \in H$ and $k \in K$, even though the elements of $H$ (and similarly, $K$) may not commute with each other.

**Remark 0.5.** We have essentially made $G$ from two of its subgroups. Moreover, if $G = H \otimes K$, then the previous theorem give that $G/H$ and $G/K$ are both groups. Unsurprisingly, we can show that $G/H \cong K$ and $G/K \cong H$, that is, the algebra we want to happen does happen in this scenario.

**Theorem 0.6.** Suppose $G = \langle a \rangle \otimes \langle b \rangle$ is a group such that $|a| = m$ and $|b| = n$. Then $G \cong \mathbb{Z}_m \times \mathbb{Z}_n$. 
Lemma 1: Let $G$ be a finite Abelian group of order $p^n m$, where $n \in \mathbb{N}$, $p$ is prime, and $\gcd(p, m) = 1$. Then $G = H_p \otimes K$, where $H_p = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$. Moreover, $|H_p| = p^n$.

How Does Lemma 1 Help? Suppose $G$ is an Abelian group such that $|G| = p^n m$. Then we know that $G = H_{p^1} \otimes H_{p^2} \otimes \cdots \otimes H_{p^j}$, where $|H_{p^1}| = p^{a_1^1}$ and $|H_{p^2}| = p^{a_2^2}$, for $1 \leq m \leq j$.

Lemma 2: Let $G$ be an Abelian group of order $p^n$, where $n \in \mathbb{N}$ and $p$ is prime. If $a \in G$ has maximal order, then $G = \langle a \rangle \otimes K$ for some $K \leq G$.

How Does Lemma 2 Help? This tells us how to further break up our groups of prime power orders. It also allows us to prove Lemma 3.

Lemma 3: Any Abelian group of prime power order is an internal direct product of cyclic groups.

How Does Lemma 3 Help? Once we have that $G = H_{p^1} \otimes H_{p^2} \otimes \cdots \otimes H_{p^w}$ from Lemma 1, we then know that each $H_{p^i} = \langle a_{\ell_i} \rangle \otimes \langle a_{\ell_2} \rangle \otimes \cdots \langle a_{\ell_k} \rangle$.

Lemma 4: Let $G$ be an Abelian Group and define $G^n = \{x^n \mid x \in G\}$. Then $G^n \leq G$. Moreover, if $p$ is a prime such that $p|\gcd(n, |G|)$, then $G^n \neq G$.

How Does Lemma 4 Help? We use this in the proof of Lemma 5.

Lemma 5: Suppose that $G$ is an Abelian group of order $p^n$, where $n \in \mathbb{N}$ and $p$ is prime. If $G = H_{p^1} \otimes H_{p^2} \otimes \cdots \otimes H_{p^w}$ and $G = K_{p^1} \otimes K_{p^2} \otimes \cdots \otimes K_{p^w}$, where each $H_{p_i}$ and $K_{p_i}$ is a non-trivial, cyclic subgroup of $G$ such that $|H_{p^1}| \geq |H_{p^2}| \geq \cdots \geq |H_{p^w}|$ and $|K_{p^1}| \geq |K_{p^2}| \geq \cdots \geq |K_{p^w}|$, then $w = z$ and $|H_{p^r}| = |K_{p^r}|$ for every $1 \leq r \leq w$.

How Does Lemma 5 Help? This gives us the “uniquely determined” part of the theorem.