Final Exam Review Problems

**Problem 0a:** All material from the two in-class exams is fair game for the final exam.

**Problem 0b:** Know all basic definitions (like homomorphism, factor group, and kernel) and theorems (like the Fundamental Hom. Theorem, Cauchy’s Theorem for Finite Abelian Groups, and the Fundamental Theorem of Finite Abelian Groups).

**Problem 1:** Show that no homomorphism exists from \( \mathbb{Z}_{16} \times \mathbb{Z}_2 \) onto \( \mathbb{Z}_4 \times \mathbb{Z}_4 \). Can there be a homomorphism from \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) onto \( \mathbb{Z}_8 \)? What about from \( \mathbb{Z}_{16} \) onto \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)?

**Problem 2:** For \( n \in \mathbb{N} \), how many homomorphisms exist from \( \mathbb{Z}_n \) to itself.

**Problem 3:** Show that \( (\mathbb{Z} \times \mathbb{Z})/(\langle (2, 5) \rangle) \cong \mathbb{Z} \). Is the same statement true if we replace \( \langle (2, 5) \rangle \) with \( \langle (4, 6) \rangle \)?

**Problem 4:** The converse of Lagrange’s Theorem is true for any finite Abelian group.

i. State this as a theorem, and prove, by example, that your theorem is true for the group \( \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \).

ii. Prove that every Abelian group of order \( 10m \), \( m \in \mathbb{N} \), has a cyclic subgroup of order 10.

**Problem 5:** Suppose \( f : G \to G' \) is an onto group homomorphism. Given any set \( K \subseteq G' \) define \( f^{-1}(K) = \{ x \in G \mid f(x) \in K \} \). Prove that if \( H \triangleleft G' \), then \( f^{-1}(H) \triangleleft G \).

**Problem 6:** Fix \( n \in \mathbb{N} \), and suppose \( G \) is the group of all \( n \times n \) diagonal matrices with entries of \( \pm 1 \) on the diagonal. What is the isomorphism class of \( G \)?

**Problem 7:** Prove or disprove the following statement:

*There exists an infinite group with no elements of infinite order.*