MATH 3362-1 **Modern Algebra I** **Spring 2022**

Bonus Problems

All bonus problems are due no later than 8:00 a.m. on Wednesday, May 11th.

Bonus Problem 1. (10 points) Prove that for any $n \in \mathbb{N}$, n has a unique representation of the form $n = c_1 1! + c_2 2! + \cdots + c_k k!$, for some $k \geq 1$, with integers $0 \leq c_j \leq j$ for each j and $c_k \neq 0$.
(Originally on Homework #1)

Bonus Problem 2. (10 points) Let $S = \mathbb{R} - \{-1\}$, and define $*$ on S by

$$a * b = a + b + ab$$

for any $a, b, \in S$. Prove or disprove that $(S, *)$ is a group.
(Originally on Homework #2)

Bonus Problem 3. (10 points) Let G be a group and suppose that H, K are proper subgroups of G. Prove that $G \neq H \cup K$.
(Originally on Homework #4)

Bonus Problem 4. (10 points) Let G be a group and suppose $x, y \in G$ such that $xy = yx$ and $\gcd(|x|, |y|) = 1$. Show that $|xy| = |x||y|$. Futher, find some a, b in a group H such that $\gcd(|a|, |b|) = 1$, but $|ab| \neq |a||b|$.
(Originally on Homework #4)

Bonus Problem 5. (10 points) Show that any finite group $G \neq \{e\}$ has some element of prime order.
(Originally on Homework #4)

Bonus Problem 6. (10 points) Let P be the group of all polynomials in x with coefficients in $\mathbb{Z}$ under addition. Prove that $P \approx (\mathbb{Q}^+, \cdot)$.
(Originally on Homework #7)

Bonus Problem 7. (10 points) Suppose G and H are finite groups and let

$$G_k = \{x \in G \mid |x| = k\} \text{ and } H_k = \{x \in H \mid |x| = k\}.$$

Find, with proof, G and H such that $|G_k| = |H_k|$ for every possible value of k, but $G \not\approx H$?
(Originally on Homework #7)