Matrices & **Linear Algebra**

Definition: $\mathbf{A} = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is an $m \times n$ real matrix if a_{ij} is real for every i and j.

Example:

$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & -7 & 3 & \pi \end{pmatrix}$ is a 2×4 real matrix.

$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 2 & \sqrt{2} \\ 3 & 0 \end{pmatrix}$ is a 3×2 real matrix.

Theorem: Two matrices are equal if and only if all of their entries are identical.

Matrix Algebra

1. Scalar Multiplication

Let c be any real number. Then $c\mathbf{A} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}$.

Example

Consider $\mathbf{P}$ from above. We have that $2\mathbf{P} = \begin{pmatrix} 2 & 2 & 0 & 4 \\ 4 & -14 & 6 & 2\pi \end{pmatrix}$.

2. Matrix Addition

Note: If $\mathbf{A}$ is an $m \times n$ matrix, then you can only add it to an $m \times n$ matrix $\mathbf{B}$. The resulting matrix, $\mathbf{A} + \mathbf{B}$, will be an $m \times n$ matrix.

Example

Let $\mathbf{R} = \begin{pmatrix} 1 & 3 \\ 1 & -6 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} -2 & 0 \\ 3 & 10 \end{pmatrix}$. Then

$$\mathbf{R} + \mathbf{S} = \begin{pmatrix} 1 + (-2) & 3 + 0 \\ 1 + 3 & (-6) + 10 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 4 & 4 \end{pmatrix}.$$

3. Matrix Multiplication

Note: If $\mathbf{A}$ is an $m \times n$ matrix, then you can only multiply it by an $n \times p$ matrix $\mathbf{B}$. The resulting matrix, $\mathbf{AB}$, will be an $m \times p$ matrix. This will most likely not be equal to the matrix $\mathbf{BA}$, which may not even be defined.

Example

Let $\mathbf{C} = \begin{pmatrix} 1 & 3 & -1 \\ -1 & -2 & 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -2 & 0 \\ 3 & 10 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{CD}$ will be a 2×2 matrix. Here,

$$\begin{aligned} \mathbf{CD} &= \begin{pmatrix} (1)(-2)+(3)(3)+(-1)(0) & (1)(0)+(3)(10)+(-1)(1) \\ (-1)(-2)+(-2)(3)+(0)(0) & (-1)(0)+(-2)(10)+(0)(1) \end{pmatrix} \\ &= \begin{pmatrix} -2+9+0 & 0+30-1 \\ 2-6+0 & 0-20+0 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 29 \\ -4 & -20 \end{pmatrix}. \end{aligned}$$

So we go across the first row of $\mathbf{C}$ and down the columns of $\mathbf{D}$, until we have used up all of the columns of $\mathbf{D}$ with the first row of $\mathbf{C}$. We then repeat with the second row of $\mathbf{C}$, and so on, until all of the rows of $\mathbf{C}$ have been exhausted. Now $\mathbf{DC}$ is a 3×3 matrix (and so not equal to $\mathbf{CD}$). Here,

$$\begin{aligned} \mathbf{DC} &= \begin{pmatrix} (-2)(1)+(0)(-1) & (-2)(3)+(0)(-2) & (-2)(-1)+(0)(0) \\ (3)(1)+(10)(-1) & (3)(3)+(10)(-2) & (3)(-1)+(10)(0) \\ (0)(1)+(1)(-1) & (0)(3)+(1)(-2) & (0)(-1)+(1)(0) \end{pmatrix} \\ &= \begin{pmatrix} -2+0 & -6+0 & 2+0 \\ 3-10 & 9-20 & -3+0 \\ 0-1 & 0-2 & 0+0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -6 & 2 \\ -7 & -11 & -3 \\ -1 & -2 & 0 \end{pmatrix}. \end{aligned}$$

Example

Let $\mathbf{X} = \begin{pmatrix} -1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 5 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix}$. Then $\mathbf{YX}$ is not defined, but $\mathbf{XY}$ will be a 2×3 matrix. Here,

$$\begin{aligned} \mathbf{XY} &= \begin{pmatrix} -1+0+0+2 & 1+0-1+0 & -1+0+2+0 \\ 3-2+0+1 & -3+0+0+0 & 3+10+0+0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 2 & -3 & 13 \end{pmatrix}. \end{aligned}$$

Definition: The $n \times n$ identity matrix, denoted by $\mathbf{I}_n$, has all 1's down the main diagonal and 0's everywhere else. It has the property that for any $n \times n$ matrix $\mathbf{P}$, $\mathbf{I}_n\mathbf{P} = \mathbf{P}\mathbf{I}_n = \mathbf{P}$.

Example

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition: If $\mathbf{A}$ is an $n \times n$ matrix and if there exists another $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, then we say that $\mathbf{A}$ (and $\mathbf{B}$) is invertible. $\mathbf{B}$ is called the inverse of $\mathbf{A}$, and we denote this by $\mathbf{B} = \mathbf{A}^{-1}$. (Note: If $\mathbf{A}^{-1}$ exists, then it is unique.)

Example

Suppose $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 3 & -4 & -2 \\ 2 & -2 & 4 \end{pmatrix}$, and let $\mathbf{B} = \begin{pmatrix} 10/33 & 7/33 & -4/33 \\ 8/33 & -1/33 & -13/66 \\ -1/33 & -4/33 & 7/33 \end{pmatrix}$.

Then calculation shows that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_3$, and so $\mathbf{B} = \mathbf{A}^{-1}$.

Solving Linear Systems Using Matrices

Consider the system of equations

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 3 \\ 3x_1 - 4x_2 - 2x_3 &= 5 \\ 2x_1 - 2x_2 + 4x_3 &= 8. \end{aligned}$$

This is called a linear system of equations (because 1 is the power of each of the variables x_1, x_2, x_3), and utilizing various methods, we could solve this system to see that one solution is

$$x_1 = 1,\ x_2 = -1,\ x_3 = 1,$$

and this turns out to be the only solution. Now consider the following matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 3 & -4 & -2 \\ 2 & -2 & 4 \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.$$

Now $\mathbf{Ax} = \begin{pmatrix} 2x_1 + 2x_2 + 3x_3 \\ 3x_1 - 4x_2 - 2x_3 \\ 2x_1 - 2x_2 + 4x_3 \end{pmatrix}$, so the statement that $\mathbf{Ax} = \mathbf{b}$ is the statement that

$$\begin{pmatrix} 2x_1 + 2x_2 + 3x_3 \\ 3x_1 - 4x_2 - 2x_3 \\ 2x_1 - 2x_2 + 4x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}.$$

We can see that this, by our theorem, is equivalent to our system of equations. We call the system $\mathbf{Ax} = \mathbf{b}$ a matrix equation, and a solution to this equation is a matrix $\mathbf{x}$ (which is made up of the solutions to our system of equations). If A^{-1} exists, then one way to solve this equation is as follows:

$$\mathbf{AX} = \mathbf{b} \Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

So

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{b} &= \begin{pmatrix} (10/33)(3) + (7/33)(5) + (-4/33)(8) \\ (8/33)(3) + (-1/33)(5) + (-13/66)(8) \\ (-1/33)(3) + (-4/33)(5) + (7/33)(8) \end{pmatrix} \\ &= \begin{pmatrix} (30 + 35 - 32)/33 \\ (48 - 10 - 104)/66 \\ (-3 - 20 + 56)/33 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

Another way to solve these systems is via *Row Reduction*. As a notational device we have the *augmented matrix* $(\mathbf{A} \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 3 & -4 & -2 & 5 \\ 2 & -2 & 4 & 8 \end{array}\right)$ which is made by appending the matrix $\mathbf{b}$ to the matrix $\mathbf{A}$.

Row Reduction Rules:

There are three allowable *row operations*.

(i.) Switch two rows.
(ii.) Multiply any row by a nonzero constant.
(iii.) Add one row to another.

Example

Consider the augmented matrix $(\mathbf{A} \mid \mathbf{b})$ from the previous example. Then we have that $(\mathbf{A} \mid \mathbf{b}) = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 3 & -4 & -2 & 5 \\ 2 & -2 & 4 & 8 \end{array}\right)$. We row reduce this matrix as follows:

$$\begin{aligned}
\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 3 & -4 & -2 & 5 \\ 2 & -2 & 4 & 8 \end{array}\right) &\rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ -6 & 8 & 4 & -10 \\ -6 & 6 & -12 & -24 \end{array}\right) \\
&\rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 14 & 13 & -1 \\ 0 & 12 & -3 & -15 \end{array}\right) \\
&\rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 14 & 13 & -1 \\ 0 & 12 & -3 & -15 \end{array}\right) \\
&\rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 84 & 78 & -6 \\ 0 & -84 & 21 & 105 \end{array}\right) \\
&\rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 84 & 78 & -6 \\ 0 & 0 & 99 & 99 \end{array}\right)
\end{aligned}$$

Rewriting this new augmented matrix as a linear system, we see that we have

$$\begin{aligned}
6x_1 + 6x_2 + 9x_3 &= 9 \\
84x_2 + 78x_3 &= -6 \\
99x_3 &= 99
\end{aligned}$$

We see from the last equation that $x_3 = 1$. Substituting this value into the second equation we get that $x_2 = -1$ and substituting both of those values into the first equation we get that $x_1 = 1$. If one wished to do more row reduction, one could further row reduce the last matrix in our sequence as follows:

$$\begin{aligned}
\left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 84 & 78 & -6 \\ 0 & 0 & 99 & 99 \end{array}\right) & \rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 84 & 78 & -6 \\ 0 & 0 & -78 & -78 \end{array}\right) \\
& \rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 84 & 0 & -84 \\ 0 & 0 & -78 & -78 \end{array}\right) \\
& \rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 9 & 9 \\ 0 & 84 & 0 & -84 \\ 0 & 0 & -9 & -9 \end{array}\right) \\
& \rightarrow \left(\begin{array}{ccc|c} 6 & 6 & 0 & 0 \\ 0 & 84 & 0 & -84 \\ 0 & 0 & -9 & -9 \end{array}\right) \\
& \rightarrow \left(\begin{array}{ccc|c} -84 & -84 & 0 & 0 \\ 0 & 84 & 0 & -84 \\ 0 & 0 & -9 & -9 \end{array}\right) \\
& \rightarrow \left(\begin{array}{ccc|c} -84 & 0 & 0 & -84 \\ 0 & 84 & 0 & -84 \\ 0 & 0 & -9 & -9 \end{array}\right),
\end{aligned}$$

which reduces to

$$R = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

This leads to the system of equations

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 1. \end{aligned}$$

Of course, this system of three equations is easily solved for x_1, x_2, x_3.

Row Reduction to Find an Inverse Matrix

Suppose we are given an $n \times n$ invertible matrix $\mathbf{A}$. To find the inverse of $\mathbf{A}$, we first construct the augmented matrix $(\mathbf{A} \mid \mathbf{I}_n)$. We then row reduce $\mathbf{A}$ to $\mathbf{I}_n$. Those operations are carried over on the righthand side of the matrix as well, and whatever matrix is there once $\mathbf{A}$ has been row reduced will be the inverse of $\mathbf{A}$.

Example

Let $\mathbf{K} = \begin{pmatrix} 2 & 5 \\ 4 & 9 \end{pmatrix}$.

Then $(\mathbf{K}|\mathbf{I}_2) = \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{array}\right)$.

So,

$$\begin{aligned}
(\mathbf{K}|\mathbf{I}_2) &\rightarrow \left(\begin{array}{cc|cc} -4 & -10 & -2 & 0 \\ 4 & 9 & 0 & 1 \end{array}\right) \\
&\rightarrow \left(\begin{array}{cc|cc} -4 & -10 & -2 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \\
&\rightarrow \left(\begin{array}{cc|cc} -4 & -10 & -2 & 0 \\ 0 & 10 & 20 & -10 \end{array}\right) \\
&\rightarrow \left(\begin{array}{cc|cc} -4 & 0 & 18 & -10 \\ 0 & 10 & 20 & -10 \end{array}\right) \\
&\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -9/2 & 5/2 \\ 0 & 1 & 2 & -1 \end{array}\right),
\end{aligned}$$

and so $\mathbf{K}^{-1} = \begin{pmatrix} -9/2 & 5/2 \\ 2 & -1 \end{pmatrix}$.