**Improper Integrals Handout**

**Definition:** We say that a definite integral in which \( f(x) \) is the integrand is improper if either (a) \( f \) is discontinuous on the interval of integration or (b) the interval of integration is infinite. If in evaluating an improper integral we get a finite number, we say that the integral converges, and otherwise we say that integral diverges.

**Type I Improper Integrals**

(a) If \( f \) is continuous on \([a, b)\) but not on \([a, b]\), i.e., \( f \) is discontinuous at \( b \), then

\[
\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx
\]

if this limit exists in finite terms.

(b) If \( f \) is continuous on \((a, b]\) but not on \([a, b]\), i.e., \( f \) is discontinuous at \( a \), then

\[
\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx
\]

if this limit exists in finite terms.

(c) If \( f \) is discontinuous at \( x = c \) with \( a < c < b \), then

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \lim_{s \to c^-} \int_a^s f(x) \, dx + \lim_{t \to c^+} \int_t^b f(x) \, dx.
\]

**Important:** Notice that one limit involves \( s \) and the other \( t \). This says that this integral converges if and only if the two separate integrals converge independently of one another.

**Type II Improper Integrals**

(a) If \( \int_a^N f(x) \, dx \) exists for all numbers \( N \geq a \), then

\[
\int_a^\infty f(x) \, dx = \lim_{N \to \infty} \int_a^N f(x) \, dx.
\]

(b) If \( \int_N^a f(x) \, dx \) exists for all numbers \( N \leq a \), then

\[
\int_{-\infty}^a f(x) \, dx = \lim_{N \to -\infty} \int_N^a f(x) \, dx.
\]

(c) If \( f \) is defined for every real number \( x \), then for any real number \( a \)

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^{\infty} f(x) \, dx = \lim_{M \to -\infty} \int_M^a f(x) \, dx + \lim_{N \to \infty} \int_M^N f(x) \, dx.
\]

**Important:** This again says that this integral converges if and only if the two separate integrals converge independently of one another. It does not matter how you choose \( a \) here, and often it is chosen to be 0.