A Strategy for Testing Series for Convergence and Divergence

Suppose we are given the series $\sum a_n$. We will classify this series according to certain properties to see which tests are best fit for application. We must also consider whether or not the series we a given is positive-term or not.

1. Check to see if the terms of the sequence $\{a_n\}$ go to zero, that is, we apply the n^{th}-Term Test for Divergence to see if
$$\lim_{n\to\infty} a_n = 0.$$
If this limit is not zero then the series $\sum a_n$ diverges. If the limit is zero, then we cannot conclude anything and we need another test.

2. If $a_n = 1/n^p$ then the series is a p-series. These series converge for $p > 1$ and diverge for $p \leq 1$. If $a_n = ar^n$ then the series is a geometric series. These series converge for $|r| < 1$ and diverge if $|r| \geq 1$. Sometimes algebraic manipulations must be done to the series first to make it of the form where $a_n = ar^n$.

3. If the series has a form which is similar to that of a p-series or a geometric series, then we consider the Comparison or Limit Comparison Test. It is usually useful to apply these tests if a_n is a rational function of n or one which involves only roots of polynomials in n. Under these circumstances, one can use the Limit Comparison Test by choosing $b_n = 1/n^j$ where $j =$ (the highest power of n in the denominator) $-$ (the highest power of n in the numerator).

4. If there is a continuous, positive-valued, decreasing function $f(x)$ such that $f(n) = a_n$ for every n, and if $\int_b^\infty f(x)\,dx$ can be integrated, then the integral test can be used. For this test, it is only important that $f(x)$ is continuous and positive-valued on some interval (c, ∞) and decreasing on some interval (d, ∞), that is, the function is continuous, positive-valued, and decreasing from some point on, i.e., not necessarily on $(1, \infty)$. It is quite often the case that series involving the natural logarithm can be analyzed using this test.

5. Supposing b_n is always positive and $a_n = (-1)^{n-1}b_n$, $a_n = (-1)^n b_n$, or $a_n = (-1)^{n+1}b_n$, then the Alternating Series Test seems appropriate to try first.

6. If a_n contains factorials then use the Ratio Test. If a_n involves other products which include constants raised to a power involving n, then the Ratio Test can be used here too. The Ratio Test should rarely (as in almost never) be used for rational or algebraic functions involving polynomials in n or $\ln(n)$.

7. If $a_n = (b_n)^n$ for every n, then the Root Test should be attempted. This should also probably not be used for rational functions or natural logarithms.

8. In general, if none of these fit exactly, then you want to start eliminating tests, and quite often, the Limit Comparison Test ends up being the only test left standing. Here, you would like to choose $b_n = p(n)$ such that $p(n)$ is a geometric or p-series which *somehow* relates to a_n.