Section 7.8: Improper Integrals
Discussion Date: February 8 & 10

Definition: We say that a definite integral in which \( f(x) \) is the integrand is improper if either

i. \( f \) is discontinuous on the interval of integration, or

ii. the interval of integration is infinite.

If in evaluating an improper integral we get a finite number, we say that the integral converges, and otherwise we say that integral diverges.

Type I Improper Integrals

i. If \( f \) is continuous on \([a, b)\) but not on \([a, b]\), i.e., \( f \) is discontinuous at \( b \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to b^-} \int_{a}^{t} f(x) \, dx
\]

if this limit exists in finite terms.

ii. If \( f \) is continuous on \((a, b]\) but not on \([a, b]\), i.e., \( f \) is discontinuous at \( a \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to a^+} \int_{t}^{b} f(x) \, dx
\]

if this limit exists in finite terms.

iii. If \( f \) is discontinuous at \( x = c \) with \( a < c < b \), then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \lim_{s \to c^-} \int_{a}^{s} f(x) \, dx + \lim_{t \to c^+} \int_{t}^{b} f(x) \, dx.
\]

(Note: In iii., one limit involves \( s \) and the other \( t \). This says that this integral converges if and only if the two separate integrals converge independently of one another.)
Type II Improper Integrals

i. If $\int_a^N f(x)dx$ exists for all numbers $N \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{N \to \infty} \int_a^N f(x)dx.$$ 

ii. If $\int_N^a f(x)dx$ exists for all numbers $N \leq a$, then

$$\int_{-\infty}^a f(x)dx = \lim_{N \to -\infty} \int_N^a f(x)dx.$$ 

iii. If $f$ is defined for every real number $x$, then for any real number $a$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx = \lim_{M \to -\infty} \int_M^a f(x)dx + \lim_{N \to \infty} \int_a^N f(x)dx.$$ 

(Note: In iii., one limit involves $M$ and the other $N$. This says that this integral converges if and only if the two separate integrals converge independently of one another. It does not matter how you choose $a$ here, and often it is chosen to be 0.)