MATH 1312 | **Calculus II** | **Spring 2022**

Law of Natural Growth

Discussion Date: March 18

One model for how a population (or really, just an *amount of something*) changes over time is called the Law of Natural Growth:

A population, P(t), grows at a rate proportional to its size.

Translating this to a differential equation, where the initial population is $P_0 > 0$, gives

$$\frac{dP}{dt} = kP, P(0) = P_0.$$

By a previous problem, we know that the solution to this differential equation is

$$P(t) = Ae^{kt},$$

and solving for A we get

$$P(t) = P_0 e^k t.$$

Note that as $t \to \infty$, only two outcomes exist for $P(t)$:

1. if $k > 0$ then $P(t) \to \infty$ as $t \to \infty$; and
2. if $k < 0$ then $P(t) \to 0$ as $t \to \infty$.

Example. Suppose the rate at which a population of bacteria grows is proportional to the size of the population at that time. If we begin with a population of 200 bacteria, and 2 hours later there are 300 bacteria, how long will it take for the initial population to grow to 1000 bacteria?

Solution. Let $P(t)$ denote the population of bacteria at time $t \geq 0$, where $P(0) = 200$ and $P(2) = 300$. Now,

$$\frac{dP}{dt} = kP \Rightarrow P(t) = 200e^{kt}.$$

Solving for k, we have

$$P(2) = 200e^{2k} = 300 \Rightarrow e^{2k} = \frac{3}{2} \Rightarrow k = \frac{1}{2}\ln(3/2).$$

We now need to know when

$$P(t) = 200e^{\frac{1}{2}\ln(3/2)t} = 1000,$$

and solving for t gives that it will take $t = 2\ln(5)/\ln(3/2) \approx 7.94$ hours for the population to be 1000 bacteria.