HW #10, due November 16th

Chapter 8: 2

Extra Problems for HW #10

**Problem 1:** Recall that for any \( \alpha \in \mathbb{R} \) and any nonnegative integer \( k \),
\[
\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - (k - 1))}{k!}.
\]
Show that
\[
(-1)^k \binom{1/2}{k} = \binom{k - 3/2}{k}.
\]

**Problem 2:** Show that the Catalan numbers, \( C_n \), count the number of elements in each of the following sets. It may be easiest to give a bijection between elements from these sets and elements from a set which we know is counted by the Catalan numbers. Moreover, once you have proved one of following is counted by \( C_n \), you may then use that for future bijections.

(a) Sequences \( a_1 a_2 \cdots a_{2n} \) of \( n \) 1’s and \( n \) -1’s such that the partial sum \( a_1 + a_2 + \cdots + a_k \geq 0 \) for every \( 1 \leq k \leq 2n \).

(b) Dyck paths from \((0, 0)\) to \((2n, 0)\) such that the paths never cross the \( x \)-axis.

(c) Binary trees with \( n \) vertices.

(d) Regular \( n \)-gons which have been divided up into \( n - 2 \) triangles, where here, divisions which are oriented differently are counted as distinct.

**Problem 3:** Define \( SLP_n \) to be the set of all subdiagonal lattice paths from \((0, 0)\) to \((n, n)\) using only moves of \( N \) and \( E \), and let
\[
C_n(q) := \sum_{P \in SLP_n} q^{A(P)},
\]
where \( A(P) \) is the area between the \( x \)-axis and any \( P \in SLP_n \).

(a) Compute \( C_k(q) \) for \( k = 0, 1, 2, 3, 4 \). (Hint: They will all be polynomials in \( q \).)

(b) Show that
\[
C_4(q) = \sum_{k=0}^{3} C_k(q) C_{3-k}(q) q^{(k+1)(3-k)}.
\]

(c) For any nonnegative integer \( n \), what is the value of \( \lim_{q \to 1^+} C_n(q) \)?

**Problem 4:** Consider the \( n \times n \) matrix \( A_n \), where the entry \( i-j \)th entry of \( A_n \) is \( a_{i,j} = C_{i+j-1} \), the \((i + j - 1)^{th} \) Catalan number. For example,
\[
A_3 = \begin{pmatrix}
1 & 1 & 2 \\
1 & 2 & 5 \\
2 & 5 & 14
\end{pmatrix}.
\]
Find \( det(A_k) \) for \( k = 1, 2, 3, 4, 5 \).
**Problem 5:** Find the planar binary tree which corresponds to the following sub-diagonal lattice path sequence:

\[ EENENEENEENENNN. \]

**Problem 6:** Compute \( f^\lambda \) for \( \lambda = (6,5^2,3,2^3,1) \).

**Problem 7:** Consider \((P,Q)\) from the Robinson-Schensted algorithm.

(a) Find \((P,Q)\) corresponding to \( \sigma = (2\ 5\ 1\ 7\ 4\ 3\ 6) \) \( \in S_7 \).

(b) Find the corresponding \( \sigma \in S_7 \) if \((P,Q)\) is given by the pair below.

\[
(P= \begin{array}{ccc}
1 & 3 & 7 \\
2 & 5 \\
4 & 6 \\
\end{array} , \quad Q= \begin{array}{ccc}
1 & 2 & 6 \\
3 & 5 \\
4 & 7 \\
\end{array})
\]