Chapter 8: 2

Extra Problems for HW #12

Problem 1: Recall that for any \( \alpha \in \mathbb{R} \) and any nonnegative integer \( k \),
\[
\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - (k - 1))}{k!}.
\]
Show that
\[
(-1)^k \binom{1/2}{k} = \binom{k - 3/2}{k}.
\]

Problem 2: In class it was shown that planar binary trees on \( 2n + 1 \) vertices are
in 1-1 correspondence with subdiagonal lattice paths from \((0, 0)\) to \((n, n)\). Find the
tree which corresponds to the path \( ENEENNENEEENEEENNENN \).

Problem 3: Show that the Catalan numbers, \( C_n \), count the number of elements
in each of the following sets. It may be easiest to give a bijection between elements
from these sets and elements from a set which we know is counted by the Catalan
numbers. Moreover, once you have proved one of following is counted by \( C_n \), you
may then use that for future bijections.

(a) Sequences \( a_1a_2 \cdots a_{2n} \) of \( n \) 1’s and \( n \) -1’s such that the partial sum \( a_1 + a_2 +
\cdots + a_k \geq 0 \) for every \( 1 \leq k \leq 2n \).

(b) Regular \( (n + 2) \)-gons which have been divided up into \( n \) triangles, where here,
divisions which are oriented differently are counted as distinct.

* Bonus Problem: Give a combinatorial proof different from that given in Majors’
Seminar to show that the number of 123-avoiding permutations of length \( n \) is the
\( n^{th} \) Catalan number.