Bonus Problems for Math 3343

Note: Bonus problems can be turned in for extra homework credit, but they must be worked on individually.

Bonus Problem 1: (5 points) Let \(X = \{x_1, x_2, \ldots, x_{20}\}\). Determine the number of ordered triples \((S_1, S_2, S_3)\) such that
\[S_1 \cup S_2 \cup S_3 = X,\]
and
\[S_1 \cap S_2 \cap S_3 = \emptyset.\]

Bonus Problem 2: (8 points) Give a combinatorial proof of the following identity:
\[(n + 1)n2^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k}.\]

Bonus Problem 3: (5 points) Let \(n \geq 0\). Find a formula for the following sum, and give a combinatorial proof that your formula is correct. Use your formula to compute \(S(10)\) when \(m_1 = 5\), \(m_2 = 7\), and \(m_3 = 10\).
\[S(n) = \sum_{r,s,t \geq 0, r+s+t=n} \binom{m_1}{r} \binom{m_2}{s} \binom{m_3}{t}\]

Bonus Problem 4: (8 points) Joe thinks he’s watching too much t.v., so he decides to limit his usage. He decides that for the next 7 weeks, he will watch at least 1 hour of t.v. per day but no more than 11 hours in any calendar week, and he will only watch t.v. for a whole number of hours each day. Show that there is some consecutive string of days where Joe watches exactly 20 hours of television.

Bonus Problem 5: (8 points) For any sequence of real numbers \(a_1, a_2, \ldots, a_{mnp+1}\), show that there either exists an increasing subsequence of length \(m+1\), a decreasing subsequence of length \(n+1\), or a constant subsequence of length \(p+1\), where \(m, n, p \in \mathbb{N}\).

Bonus Problem 6: (8 points) Without help from the Internet, textbooks, etc., show that \(R(3, 3, 3) = 17\).

Bonus Problem 7: (8 points) Let \(a_n\) denote the number of ways to color the cells of an \(n \times 1\) board with the colors red, blue, and green such that the number of red cells is even and there is at least one blue cell. Defining \(a_0 = 1\), find a recurrence relation for the sequence \(a_n\).
**Bonus Problem 8:** (8 points) Give a combinatorial proof of the following identity:

\[ f_{2n+1} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{i} \binom{n-j}{j}. \]

**Bonus Problem 9:** (12 points) Give a combinatorial proof of the value of

\[ \xi(n) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} r_{n-k}(B_n) f_{k-j}(B_k). \]

**Bonus Problem 10:** (12 points) In class we gave a “pictorial bijection” to show that \( S(n, k) = r_{n-k}(B_n) \). Give a similar bijection to show that \( c(n, k) = f_{n-k}(B_n) \).

**Bonus Problem 11:** (12 points) Given a permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) of \([n]\), we say \( \sigma \) is 123-avoiding if there do not exist \( i < j < k \) such that \( \sigma_i < \sigma_j < \sigma_k \). Give a combinatorial proof to show that the number of 123-avoiding permutations of \([n]\) is the \( n^{th} \) Catalan number. As an example, 41532 is 123-avoiding but 41325 is not.