Problem 1: Give a combinatorial proof that \( f_{n+2} - 1 = \sum_{k=0}^{n} f_k \).

Problem 2: Let \( a_n \) denote the ways to color the cells of the \( n \times 1 \) board with the colors red, blue, and green such that no two adjacent cells may be colored red. Assuming \( a_0 = 1 \), find a recurrence relation for \( a_n \) and give justification that your relation is correct.

Problem 3: Show that \( f_{n+1} \) counts the number of subsets of \([n]\) such that the subsets do not contain consecutive integers.

Problem 4: Find the number of \( n \)-length binary sequences \((x_1, x_2, x_3, \ldots, x_n)\) such that \( x_1 \leq x_2 \geq x_3 \leq \cdots \).

Problem 5: Show that \( f_n f_{n+1} = \sum_{k=0}^{n} f_k^2 \).

Problem 6: Without rewriting the terms using the recursion, give a combinatorial proof that \( 2f_n = f_{n+1} + f_{n-2} \).

Problem 7: Show that for \( n \geq 2 \), \( f_{n-2} \) counts the tilings of the \( n \times 1 \) board using tiles of length 2 or greater.

Problem 8: Show that \( f_{2n+1} = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} \).