Problem 1: If $c(n, k)$ is defined to be the number of permutations of $[n]$ which contain exactly $k$ nonempty cycles with $c(0, 0) = 1$ and $c(n, k) = 0$ whenever $k > n$ or $n, k < 0$, then give a combinatorial proof that $c(n, k)$ satisfies the recursion

$$c(n + 1, k) = c(n, k - 1) + nc(n, k).$$

Problem 2: Verify that the relationship given in class between proper lattice paths from $(0, 0)$ to $(n, n)$ which cross the diagonal and sequences of $n + 1$ E’s and $n - 1$ N’s is a bijection.

Problem 3: In class we defined a function $\psi$ that mapped labeled, planar binary trees on $2n + 1$ vertices into proper, subdiagonal lattice paths from $(0, 0)$ to $(n, n)$.

i. Show that $\psi$ is indeed bijection, as claimed in class.

ii. Using this map, find the labeled, planar binary tree which corresponds to the path $ENEENNENENEEENNEENN$. 

Problem 4: For any $\alpha \in \mathbb{R}$ and any $k \in \mathbb{N}$, the generalized binomial coefficient is

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - (k - 1))}{k!}.$$

Show that $(-1)^k \binom{\frac{1}{2}}{k} = \binom{k - \frac{3}{2}}{k}$. This is a computational exercise, and so do not attempt to give a combinatorial proof.

Problem 5: Show that the Catalan numbers, $C_n$, count the number of elements in each of the following sets. It may be easiest to give a bijection between elements from these sets and elements from a set which we know is counted by the Catalan numbers. Moreover, once you have proved one of following is counted by $C_n$, you may then use that for future bijections.

i. Sequences $a_1 a_2 \ldots a_{2n}$ of $n$ 1’s and $n$ -1’s such that the partial sum $a_1 + a_2 + \cdots + a_k \geq 0$ for every $1 \leq k \leq 2n$.

ii. $2 \times n$ arrays, $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$, filled with the numbers $1, 2, \ldots, 2n$ such that for all $i$, $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $a_i < b_i$.

* Bonus Problem: Given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ of $[n]$, we say $\sigma$ is 123-avoiding if there do not exist $i < j < k$ such that $\sigma_i < \sigma_j < \sigma_k$. Give a combinatorial proof to show that the number of 123-avoiding permutations of $[n]$ is the $n^{th}$ Catalan number. As an example, 41532 is 123-avoiding but 41325 is not.