Recall that $n!, P(n, k), \binom{n}{k}, f_n, S(n, k), D_n,$ etc. are acceptable symbols to have in solutions.

**Problem 0.** (From the book.) Read Sections 4.1, 5.3, 8.1, and 8.2 and do the following problems:
- Problem Set 14: 5-7
- Problem Set 20: 2
- Problem Set 28: 4

**Problem 1.** Given two permutations of $[n]$, say $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and $\tau = \tau_1\tau_2\cdots\tau_n$, we say that $\sigma$ and $\tau$ are incompatible if $\sigma_i \neq \tau_i$ for each $1 \leq i \leq n$. Then a derangement of $[n]$ is simply a permutation that is incompatible with the identity permutation.

i. Suppose $\alpha \in S_n$ and define $D_\alpha = \{\sigma \in S_n \mid \sigma \text{ is incompatible with } \alpha\}$. Show that $|D_\alpha| = D_n$.

ii. In class we defined the sets

\[
D_{n,k} = \{\sigma \in S_n \mid \sigma \text{ is a derangement of } [n] \text{ and } \sigma(1) = k\},
\]

\[
A = \{\text{derangements of } [n] \text{ of the form } \sigma = 2i_3\cdots i_n\},
\]

\[
B = \{\text{derangements of } [n] \text{ of the form } \sigma = 2i_2i_3\cdots i_n \text{ with } i_2 \neq 1\}.
\]

Use part i. to show that $|D_{n,i}| = |D_{n,j}|$ for all $i \neq j$, $|A| = D_{n-2}$, and $|B| = D_{n-1}$.

**Problem 2.** Let $S(n, k)$ denote the Stirling number of the second kind, and assume that $0 \leq k \leq n + 1$.

i. Prove that $S(n, k)$ is the number of unordered set partitions of an $n$-element set into $k$ nonempty parts.

ii. Show that for $n \geq 2$, $S(n, 2) = 2^{n-1} - 1$. 
Problem 3. At a wine tasting party, there are 20 people who each bring a bottle of wine to sample. Assume that a tasting round consists of every partygoer sampling a single wine, but nobody sampling the same wine as somebody else. If there are two tasting rounds such that nobody can sample a wine they have previously sampled, how many possible ways are there for the 20 people to accomplish the two rounds of tasting? Here, assume that it matters which round a person tasted a wine, i.e., Alice drinking Wine X in Round 1 and Wine Y in Round 2 is different than her drinking Wine Y in Round 1 and Wine X in Round 2.

Problem 4. Determine the number of permutations of [9] with the following property:

i. No odd integer is in its natural position.

ii. Exactly two odd integers are in the natural position.